# §4. A GENERALIZATION OF THE INVARIANCE OF

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## §3. THE FOUR VERTEX THEOREM IN $\mathbf{R}^2$

Let $\gamma$ be a closed embedded curve on $\mathbf{R}^2$. The Euclidean curvature $\kappa$ is defined and so it must have a minimum and a maximum which give two vertices on $\gamma$. (Indeed the number of local minima must be the same as the number of local maxima, so that the number of extrema is even.) Next we move $\gamma$ by a Möbius transformation so as to send one of these extrema to $\infty$, and in such a way that the curve becomes asymptotic to the $x$-axis. Now $\kappa(s) \to 0$ as $s \to \pm\infty$, and the theorem of turning tangents ([6], p. 37) says that $\int \kappa(s)\, ds = 0$. It follows that $\kappa$ cannot have just one maximum or just one minimum for if so it would have a fixed sign and then the integral could not be zero. Thus $\gamma$ has at least 2 extrema in addition to the one at infinity. But since the total number of extrema is even, there must be at least four of them, and hence four vertices.

There is a subtle point which we have glossed over in this argument. The vertices come in two types. As well as the extrema of $\kappa$ (the "honest" vertices) there may also be non-extremal critical points of $\kappa$. The above "proof" has used the fact that not only are the vertices inversive invariants, but so too are the isolated extrema. Whereas this is indeed true (as is implied by equation 4.3 of the next section), it suffices to note that the non-extremal critical points of $\kappa$ are unstable phenomena and each of them may be eliminated by a deformation of the curve with support in a small neighborhood of it. One may thus assume that all of the vertices of $\gamma$ are extrema, whereupon the above proof stands as is.

*Remark* 3.1. The reader may compare the above proof to that of [10], where the four vertex theorem is obtained by using a Möbius transformation to send a *non*-vertex point to infinity.

## §4. A GENERALIZATION OF THE INVARIANCE OF $\omega$

Let $(M, h)$ be a Riemannian surface with metric $h$, and let $\gamma$ be a curve on $M$ with geodesic curvature $\kappa_g$. We can ask whether the 1-form along a curve $\gamma$ given by

$$\omega_\gamma = \sqrt{|\kappa_g'|}\, ds$$

is a conformal invariant. More precisely, let $\Psi : (M_1, h_1) \to (M_2, h_2)$ be a conformal map and let $\gamma_1$ be a curve on $M_1$; is it true that

$$\Psi^*(\omega_{\Psi(\gamma_1)}) = \omega_{\gamma_1} \quad . \tag{4.1}$$

A necessary condition for this is that $\Psi$ send curves of constant geodesic curvature to curves of constant geodesic curvature. We show that this condition is in fact sufficient.

THEOREM 4.2. *Let* $\Psi:(M_1, h_1) \to (M_2, h_2)$ *be a conformal map which sends curves of constant geodesic curvature to curves of constant geodesic curvature. Then 4.1 holds for all curves in* $M_1$.

*Proof.* Let $\gamma_1$ be a curve on $M_1$ parametrized by arc-length $s_1$ and with geodesic curvature $\kappa_1(s_1)$; similarly let $\gamma_2 = \Psi(\gamma_1)$ be the image curve on $M_2$ parametrized by arc-length $s_2$ with geodesic curvature $\kappa_2(s_2)$. We must show that

$$\Psi^*\left(\sqrt{|\kappa_2'(s_2)|}\, ds_2\right) = \sqrt{|\kappa_1'(s_1)|}\, ds_1 \,.$$

We will prove this for orientation reversing conformal maps $\Psi$; the general case then follows by composition. We start with a technical lemma.

LEMMA. *If* $p \in \gamma_1$ *then there exists a neighborhood* $\Omega$ *of* $x$ *in* $M_1$ *and a positively oriented orthonormal frame* $(\tau, \mathbf{n})$ *of vector fields on* $\Omega$ *such that the following properties hold.*

a) *The connected component containing* $x$ *of the intersection of* $\gamma_1$ *with* $\Omega$ *is a flow line for* $\tau$.

b) *The flow lines of* $\mathbf{n}$ *have constant curvature.*

*Proof of Lemma.* It suffices to take a sufficiently small neighborhood $\Omega$ of $p$ such that there exist on it a flow $\mathbf{n}$ by geodesics which are perpendicular to the connected component of $\gamma_1 \cap \Omega$ containing $x$. Then $\tau$ is chosen perpendicular to $\mathbf{n}$.

Returning to the proof of the theorem, it obviously suffices to work on the neighborhood $\Omega$ of some arbitrary point $x$ of $M_1$. Since $\Psi$ is conformal there exists a positive function $f$ on $\Omega$ such that $(\Psi_*(f\tau), -\Psi_*(f\mathbf{n}))$ is a positively oriented orthonormal framing of $\Psi(\Omega)$, where $\Psi_*$ is the push-forward map on tangent vectors induced by $\Psi$. We get $g_1 = f^2\Psi^*(g_2)$, where $\Psi^*(g_2)$ is the pull-back of $g_2$ by $\Psi$. Now

$$\Psi^*\left(\sqrt{|\kappa_2'(s_2)|}\, ds_2\right) = f^{-1}\Psi^*\left(\sqrt{|\kappa_2'(s_2)}\right)\, ds_1$$

so it suffices to show that

$$\Psi^*\left(\kappa_2'(s_2)\right) = -f^2\kappa_1'(s_1)\,. \tag{4.3}$$

By the lemma we can take $\tau = d/ds_1$ on $\gamma_1$ so that

$$\kappa_1'(s_1) = \tau(\kappa_1), \quad \text{and}$$
$$\Psi^*(\kappa_2'(s_2)) = \Psi^*(\Psi_*(f\tau)\kappa_2) = f\tau(\Psi^*(\kappa_2))$$

and hence it suffices to show that

$$\tau(\Psi^*(\kappa_2)) = -f\tau(\kappa_1) \tag{4.4}$$

Now the curvature $\kappa_1$ is given by the standard formula $\kappa_1 = -g_1(\nabla_\tau \mathbf{n}, \tau) = g_1([\mathbf{n}, \tau], \tau)$, where $\nabla$ is the Levi-Civita connection and $[\mathbf{n}, \tau]$ is the Lie bracket of vector fields $\mathbf{n}$ and $\tau$. Similarly the curvature $\kappa_2$ is given by

$$\kappa_2 = g_2([-\Psi_*(f\mathbf{n}), \Psi_*(f\tau)], \Psi_*(f\tau)) = g_2(\Psi_*[f\tau, f\mathbf{n}], \Psi_*(f\tau))$$

and therefore

$$\begin{aligned} \Psi^*(\kappa_2) &= f^{-2}g_1([f\tau, f\tau) \\ &= f^{-2}g_1(f^2[\tau, \mathbf{n}] + f\tau(f)\mathbf{n} - f\mathbf{n}(f)\tau, f\tau) \\ &= -f\kappa_1 - \mathbf{n}(f) \end{aligned} \tag{4.5}$$

Thus, in order to prove 4.4, we will show that

$$\begin{gathered} \tau(f\kappa_1 + \mathbf{n}(f)) = f\tau(\kappa_1) \\ \text{i.e.} \quad \kappa_1\tau(f) + \tau(\mathbf{n}(f)) = 0\,. \end{gathered} \tag{4.6}$$

To do this write $[\mathbf{n}, \tau]$ as a linear combination of $\tau$ and $\mathbf{n}$

$$[\mathbf{n}, \tau] = \kappa_1\tau + \mu_1\mathbf{n} \tag{4.7}$$

where of course $\mu_1$ is the geodesic curvature of the flow lines of $\mathbf{n}$. If $\mu_2$ is the geodesic curvature of the flow lines of $\Psi_*(\mathbf{n})$, then analogously to 4.5 we have

$$\Psi^*(\mu_2) = -f\mu_1 + \tau(f)\,. \tag{4.8}$$

Since the flow lines of $\mathbf{n}$ have constant curvature we have

$$\mathbf{n}(\mu_1) = 0 \tag{4.9}$$

Since $\Psi$ sends curves of constant curvature to curves of constant curvature we also have

$$\mathbf{n}(\Psi^*(\mu_2)) = \Psi^*(\Psi_*(\mathbf{n})\mu_2) = 0 \tag{4.10}$$

Applying $\mathbf{n}$ to 4.8 yields, in view of 4.9) and 4.10)

$$\mathbf{n}(f)\mu_1 = \mathbf{n}\tau(f)$$

Combining this with 4.5 yields

$$\mathbf{n}(\tau(f)) - \tau(\mathbf{n}(f)) = [\mathbf{n}, \tau](f) = \kappa_1\tau(f) + \mu_1\mathbf{n}(f) = \kappa_1\tau(f) + \mathbf{n}\tau(f)$$

which gives 4.6 as required.

## §5. A GENERALIZED FOUR VERTEX THEOREM

The curves of constant curvature in the round 2-sphere $S^2$, the upper half plane $H^2$ (hyperbolic space), and the Euclidean plane $\mathbf{R}^2$ are just the circles. Moreover, the stereographic projection $p: S^2 \to \mathbf{R}^2$ and the inclusion $i: H^2 \to \mathbf{R}^2$ both preserve these circles. Thus theorem 4.2 says that our form

$$\omega = \sqrt{|\kappa'(s)|}\, ds$$

along a curve $\gamma$ in $\mathbf{R}^2$ pulls back via $p$ or $i$ to the form

$$\omega = \sqrt{|\kappa_g'(s)|}\, ds$$

along the corresponding curve $\gamma'$, where here $\kappa_g(s)$ and $s$ refer to the geodesic curvature and arc-length of $\gamma'$ in the metric for $S^2$ or $H^2$. Thus we obtain the four vertex theorem for $S^2$ and $H^2$. It follows that the four vertex theorem holds for all complete simply connected Riemannian surfaces of constant curvature. Finally if $\gamma$ is a null-homotopic smooth simple closed curve on an arbitrary complete Riemannian surface $M$ of constant curvature, then $\gamma$ lifts one-to-one to a smooth simple closed curve with the same number of vertices on the simply connected universal cover of $M$. Once again it follows that the number of vertices is at least four.

*Remark* 5.1. Interestingly, simple closed homotopically non-trivial curves in the real projective plane always have at least three vertices [17]. Note that in non-orientable surfaces the number of honest vertices of a closed curve need not necessarily be even, since here geodesic curvature is only defined up to a sign.

## §6. NORMAL FORM AND INVERSIVE CURVATURE

Let $p$ be a non-vertex point of an oriented curve $\gamma$. Since the subgroup of Euclidean motions in $G$ acts transitively on the points of $\mathbf{R}^2$ and the unit tangent vectors at these points, we may assume that the point $p \in \gamma$ which