2. Lang-Siegel towers

in $X'(S)$. Suppose X is an Abelian scheme over S and R is the subgroup of $X(S)$ consisting of constant sections of X/S. Let $s \in X(S)$. Then the set $s + R$ is a set of bounded height.

LEMMA 3.1.1 (Manin). *Suppose E is a finite dimensional K vector subspace of $K(C)$. Then the set*

$$T = \{s \in C(S) \colon \exists\, k \neq 0 \in E \quad \text{such that} \quad s^*k = 0\}$$

has bounded height.

Proof. Without loss of generality we may increase E to suppose that the rational map $g\colon C \to \mathbf{P}_K(E)$ given on points by $x \to (e \in E \to e(x))$ is birational onto its image (note: g is actually a morphism on the compliment of the polar locus of E). It follows that g induces an embedding of the generic fiber of C/S into $\mathbf{P}_{K(S)}(E \otimes K(S))$. Let h denote the logarithmic height with respect to this embedding. It follows that if $s \in C(S)$, $g \circ s$ is constant or $g \circ s$ has degree one. In the former case $h(s)$ is zero and the degree of the Zariski closure of $g \circ s(S)$ in $\mathbf{P}(E)$ in the latter.

Now if $s \in T$, and $g \circ s$ is not constant, it follows that the Zariski closure of $g \circ s(S)$ is a component of a hyperplane section of the Zariski closure of $g(C)$. Hence, $h(s)$ is less than or equal to the degree of the Zariski closure of $g(C)$. This proves the lemma. □

The key property about heights we will need is:

THEOREM 3.1.2. *Suppose $C \to S$ is as in the above theorem. If $C(S)$ contains an infinite set of bounded height then C is a constant family.*

(See Corollary 2.2, Chapter 8 of [L-FD].)

Hence all we need prove is that the elements of $C(S)$ have bounded height.

2. LANG-SIEGEL TOWERS

Suppose the genus of C is at least 1. Suppose T is an infinite subset of $C(S)$.

PROPOSITION 3.2.1. *There exists a projective system of curves*

$(\{C_n\}, \{h_{m,n}\}), m, n \in \mathbf{Z}_{>0}$ *and* $n \leqslant m$, *over* K *such that*

(i) $C_1 = C$,

(ii) $h_{m,n}\colon C_m \to C_n$ *is étale,*

(iii) $(h_{m,1})^{-1}(T) \cap C_m(S)$ *is infinite,*

(iv) *There exists a finite covering* $S_{m,n}$ *of* S *such that the fiber product of* $h_{m,n}$ *with* $S_{m,n}$ *is Galois, Abelian and of positive degree.*

Let J denote the Jacobian scheme of C over S. Let $a: C \to J$ be an Albanese morphism. Let p be a prime. Let $\bar{T}$ denote the closure of $a(T)$ in $J(S) \otimes \mathbf{Z}_p$. Since $a(T)$ is infinite it follow from the Mordell-Weil Theorem that there exists a $t \in \bar{T} - a(T)$. Let $t_n \in T$ such that $t - a(t_n) \in p^n J(S)$. Let C_n denote the normalization of the fiber-product of C and J via the map $H_n: x \to p^n x + t_n$ and $h_{n,1}$ the natural map from C_n to C. It follows that C_n is defined over S and since $H_m(J(S)) \supseteq \{t_n : m \mid n\}$ that $h_{n,1}(C_m(S))$ contains an infinite subset of T.

All that remains is to exhibit the maps $h_{m,n}$. Clearly, $t_m - t_n = p^n r_{m,n}$ for some $r_{m,n} \in J(S)$. Let $H_{m,n}$ denote the map $x: p^{m-n}x + r_{m,n}$. Then $H_{m,k} = H_{n,k} \circ H_{m,n}$. It follows that $H_{m,n}$ pulls back to a morphism $h_{m,n}: C_m \to C_n$. It is easy to see that this morphism becomes Abelian after adjoining the p^{m-n}-torsion points on J. This proves the proposition. □

Remark. One can also prove the above proposition with the condition $n \leqslant m$ replaced by $n \mid m$.

3. Corollaries of the Theorem of the Kernel

LEMMA 3.3.1. *Suppose* $g: X' \to X$ *is a morphism of smooth proper schemes with geometrically connected fibers over* S. *Then if* $\mu \in PF(X'/S)$ *and* $s, t \in X(S)$, $(g^*\mu)(s,t) = \mu(g \circ s, g \circ t)$.

Proof. This follows easily from Lemma 1.3.2. □

Suppose J is the Jacobian of C over S and g is an Albanese morphism, then since $g^*: H^1_{DR}(J/S) \to H^1_{DR}(C/S)$ is an isomorphism $g^*: PF(J/S) \to PF(C/S)$ is an isomorphism.

LEMMA 3.3.2. *Let* μ *be a fixed Picard-Fuchs differential equation on* C/S. *Then* $\{\mu(s,t): s, t \in C(S)\}$ *lies in a finite dimensional subspace of* $K[S]$ *over* K.

Proof. Suppose $\tilde{\mu} \in PF(J/S)$ such that $g^*\tilde{\mu} = \mu$. The lemma follows from the Mordell-Weil theorem which together with the Theorem of the kernel implies that $J(S)$ modulo the kernel of the homomorphism $s \to \tilde{\mu}(e, s)$ is a finitely generated Abelian group. □