

2. Lang-Siegel towers

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in $X'(S)$. Suppose X is an Abelian scheme over S and R is the subgroup of $X(S)$ consisting of constant sections of X/S . Let $s \in X(S)$. Then the set $s + R$ is a set of bounded height.

LEMMA 3.1.1 (Manin). *Suppose E is a finite dimensional K vector subspace of $K(C)$. Then the set*

$$T = \{s \in C(S) : \exists k \neq 0 \in E \text{ such that } s*k = 0\}$$

has bounded height.

Proof. Without loss of generality we may increase E to suppose that the rational map $g: C \rightarrow \mathbf{P}_K(E)$ given on points by $x \rightarrow (e \in E \rightarrow e(x))$ is birational onto its image (note: g is actually a morphism on the complement of the polar locus of E). It follows that g induces an embedding of the generic fiber of C/S into $\mathbf{P}_{K(S)}(E \otimes K(S))$. Let h denote the logarithmic height with respect to this embedding. It follows that if $s \in C(S)$, $g \circ s$ is constant or $g \circ s$ has degree one. In the former case $h(s)$ is zero and the degree of the Zariski closure of $g \circ s(S)$ in $\mathbf{P}(E)$ in the latter.

Now if $s \in T$, and $g \circ s$ is not constant, it follows that the Zariski closure of $g \circ s(S)$ is a component of a hyperplane section of the Zariski closure of $g(C)$. Hence, $h(s)$ is less than or equal to the degree of the Zariski closure of $g(C)$. This proves the lemma. \square

The key property about heights we will need is:

THEOREM 3.1.2. *Suppose $C \rightarrow S$ is as in the above theorem. If $C(S)$ contains an infinite set of bounded height then C is a constant family.*

(See Corollary 2.2, Chapter 8 of [L-FD].)

Hence all we need prove is that the elements of $C(S)$ have bounded height.

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Suppose the genus of C is at least 1. Suppose T is an infinite subset of $C(S)$.

PROPOSITION 3.2.1. *There exists a projective system of curves*

$(\{C_n\}, \{h_{m,n}\})$, $m, n \in \mathbf{Z}_{>0}$ and $n \leq m$, over K such that

- (i) $C_1 = C$,
- (ii) $h_{m,n}: C_m \rightarrow C_n$ is étale,

- (iii) $(h_{m,1})^{-1}(T) \cap C_m(S)$ is infinite,
- (iv) There exists a finite covering $S_{m,n}$ of S such that the fiber product of $h_{m,n}$ with $S_{m,n}$ is Galois, Abelian and of positive degree.

Let J denote the Jacobian scheme of C over S . Let $a: C \rightarrow J$ be an Albanese morphism. Let p be a prime. Let \bar{T} denote the closure of $a(T)$ in $J(S) \otimes \mathbf{Z}_p$. Since $a(T)$ is infinite it follows from the Mordell-Weil Theorem that there exists a $t \in \bar{T} - a(T)$. Let $t_n \in T$ such that $t - a(t_n) \in p^n J(S)$. Let C_n denote the normalization of the fiber-product of C and J via the map $H_n: x \rightarrow p^n x + t_n$ and $h_{n,1}$ the natural map from C_n to C . It follows that C_n is defined over S and since $H_m(J(S)) \supseteq \{t_n: m \mid n\}$ that $h_{n,1}(C_m(S))$ contains an infinite subset of T .

All that remains is to exhibit the maps $h_{m,n}$. Clearly, $t_m - t_n = p^n r_{m,n}$ for some $r_{m,n} \in J(S)$. Let $H_{m,n}$ denote the map $x: p^{m-n}x + r_{m,n}$. Then $H_{m,k} = H_{n,k} \circ H_{m,n}$. It follows that $H_{m,n}$ pulls back to a morphism $h_{m,n}: C_m \rightarrow C_n$. It is easy to see that this morphism becomes Abelian after adjoining the p^{m-n} -torsion points on J . This proves the proposition. \square

Remark. One can also prove the above proposition with the condition $n \leq m$ replaced by $n \mid m$.

3. COROLLARIES OF THE THEOREM OF THE KERNEL

LEMMA 3.3.1. Suppose $g: X' \rightarrow X$ is a morphism of smooth proper schemes with geometrically connected fibers over S . Then if $\mu \in PF(X'/S)$ and $s, t \in X(S)$, $(g^*\mu)(s, t) = \mu(g \circ s, g \circ t)$.

Proof. This follows easily from Lemma 1.3.2. \square

Suppose J is the Jacobian of C over S and g is an Albanese morphism, then since $g^*: H_{DR}^1(J/S) \rightarrow H_{DR}^1(C/S)$ is an isomorphism $g^*: PF(J/S) \rightarrow PF(C/S)$ is an isomorphism.

LEMMA 3.3.2. Let μ be a fixed Picard-Fuchs differential equation on C/S . Then $\{\mu(s, t): s, t \in C(S)\}$ lies in a finite dimensional subspace of $K[S]$ over K .

Proof. Suppose $\tilde{\mu} \in PF(J/S)$ such that $g^*\tilde{\mu} = \mu$. The lemma follows from the Mordell-Weil theorem which together with the Theorem of the kernel implies that $J(S)$ modulo the kernel of the homomorphism $s \rightarrow \tilde{\mu}(e, s)$ is a finitely generated Abelian group. \square