

1. Sets of bounded height

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III. MORDELL'S CONJECTURE

Suppose L is a field of characteristic zero of finite type over a relatively algebraically closed subfield K .

THEOREM 3.1 (Manin). *Suppose C is a curve of genus at least 2 defined over K . Suppose $C(L)$ is infinite, then there exists a curve C_0 defined over K such that $C_0 \times_K L \cong C$ and $C(K)$ minus the image of $C_0(K)$ under this isomorphism is finite.*

We can translate this into

THEOREM 3.1 (BIS). *Suppose S is a variety defined over K and suppose $C \rightarrow S$ is a smooth proper curve of genus at least 2 over S . Suppose $C(S)$ is infinite, then there exists a curve C_0 defined over K such that $C_0 \times_K S \cong C$ and $C(S)$ minus the image of $C_0(K)$ under this isomorphism is finite.*

Remarks. First, it is possible to reduce this by standard arguments to the case in which S is a smooth affine curve over K and so we will suppose this to be the case. Second, if we can prove that $C_0 \times_K X \cong C$ for some C_0 defined over K , (i.e. that C is a constant family) then this is de Franchis' theorem which is proven in Lang's *Fundamentals of Diophantine Geometry*. Hence to prove this theorem all we have to do is show that if $C(S)$ is infinite then C is a constant family of curves.

1. SETS OF BOUNDED HEIGHT

In this section we will either recall or derive the properties of heights needed in the sequel.

Let $f: X \rightarrow S$ be a smooth projective morphism of varieties over K a field of characteristic zero. Corresponding to a projective embedding of X over S , there exists a function $h: X(S) \rightarrow \mathbf{R}$ called a logarithmic height. (For a reference, see ([L-FD] Chapter 3, §3). If the logarithmic height of a subset of $X(S)$ is bounded with respect to one projective embedding, it is bounded with respect to all (See [L] Prop. 1.7, Chapt. 4). We will call such a set a set of bounded height and a set of points which is not of bounded height, a set of unbounded height. We will need several properties of such sets. If $g: X' \rightarrow X$ is a morphism of projective schemes over S which is finite onto its image, then the inverse image of a set of bounded height in $X(S)$ is a set of bounded height

in $X'(S)$. Suppose X is an Abelian scheme over S and R is the subgroup of $X(S)$ consisting of constant sections of X/S . Let $s \in X(S)$. Then the set $s + R$ is a set of bounded height.

LEMMA 3.1.1 (Manin). *Suppose E is a finite dimensional K vector subspace of $K(C)$. Then the set*

$$T = \{s \in C(S) : \exists k \neq 0 \in E \text{ such that } s*k = 0\}$$

has bounded height.

Proof. Without loss of generality we may increase E to suppose that the rational map $g: C \rightarrow \mathbf{P}_K(E)$ given on points by $x \rightarrow (e \in E \rightarrow e(x))$ is birational onto its image (note: g is actually a morphism on the complement of the polar locus of E). It follows that g induces an embedding of the generic fiber of C/S into $\mathbf{P}_{K(S)}(E \otimes K(S))$. Let h denote the logarithmic height with respect to this embedding. It follows that if $s \in C(S)$, $g \circ s$ is constant or $g \circ s$ has degree one. In the former case $h(s)$ is zero and the degree of the Zariski closure of $g \circ s(S)$ in $\mathbf{P}(E)$ in the latter.

Now if $s \in T$, and $g \circ s$ is not constant, it follows that the Zariski closure of $g \circ s(S)$ is a component of a hyperplane section of the Zariski closure of $g(C)$. Hence, $h(s)$ is less than or equal to the degree of the Zariski closure of $g(C)$. This proves the lemma. \square

The key property about heights we will need is:

THEOREM 3.1.2. *Suppose $C \rightarrow S$ is as in the above theorem. If $C(S)$ contains an infinite set of bounded height then C is a constant family.*

(See Corollary 2.2, Chapter 8 of [L-FD].)

Hence all we need prove is that the elements of $C(S)$ have bounded height.

2. LANG-SIEGEL TOWERS

Suppose the genus of C is at least 1. Suppose T is an infinite subset of $C(S)$.

PROPOSITION 3.2.1. *There exists a projective system of curves*

$(\{C_n\}, \{h_{m,n}\}), m, n \in \mathbf{Z}_{>0}$ and $n \leq m$, over K such that

- (i) $C_1 = C$,
- (ii) $h_{m,n}: C_m \rightarrow C_n$ is étale,