0. Introduction

L'Enseignement Mathématique, t. 35 (1989), p. 77-91

HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

by Beno ECKMANN

0. INTRODUCTION

0.1. We consider complex $n \times n$ — matrices $A_1, A_2, ..., A_s$, either all unitary (case U) or all orthogonal (case O); they are called Hurwitz-Radon matrices, in short HR-matrices, if

(1) $$A_j^2 = -E\,,\ A_jA_k + A_kA_j = 0\,, \quad j, k = 1, 2, ..., s\,,\ j \neq k\,;$$

E or E_n denotes the unit matrix. Such matrices are well-known to exist, even with entries $0, \pm 1, \pm i$ (case U) or $0, \pm 1$ (case O). The possible values of n are multiples $mn_0, m = 1, 2, 3, ...$ where in case $U, n_0 = 2^{s/2}$ if s is even, $n_0 = 2^{(s-1)/2}$ if s is odd. In case $O, n_0 = 2^{(s-1)/2}$ if $s \equiv 7 \bmod 8$; $n_0 = 2^{s/2}$ if $s \equiv 0, 6$; $n_0 = 2^{(s+1)/2}$ if $s \equiv 1, 3, 5$; and $n_0 = 2^{(s+2)/2}$ if $s \equiv 2, 4 \bmod 8$.

If we put $A_0 = E$ the relations (1) are equivalent to

$$f_s(x_0, x_1, ..., x_s) = \sum_0^s x_jA_j$$

being a unitary, or orthogonal respectively, matrix for all real x_j with $\sum_0^s x_j^2 = 1$. Thus f_s can be considered as a map $S^s \to U$ via $U(n)$, or $S^s \to O$ via $O(n)$ where $U = \varinjlim U(k)$ and $O = \varinjlim O(k)$ are the infinite unitary and orthogonal groups. We also write f_s for the homotopy class of that map, $f_s \in \pi_s(U)$ or $\pi_s(O)$. We recall that by the Bott periodicity theorems these groups are cyclic or 0.

THEOREM A. *If $A_1, A_2, ..., A_s$ are HR-matrices of minimal size $n = n_0(s)$ then f_s is a generator of $\pi_s(U)$, or $\pi_s(O)$ respectively, $s = 0, 1, 2, ...$.*

Remark 1. For $s = 0$ (empty set of HR-matrices) we have $f_0(x_0) = x_0(1)$, $x_0^2 = 1$; i.e., $f_0(1) = (1)$, $f_0(-1) = (-1)$, $f_0 : S^0 \to O(1) \to O$. For $s > 0$, $f_0 : S^s \to O$ clearly factors through $SO(n) \to SO$ (U being connected, the analogue is irrelevant in the unitary case).

Remark 2. The problem originally solved by Hurwitz [H] and Radon [R] refers to the case O: One asks for complex bilinear forms $z = f(x, y) = (\sum_0^s x_j A_j) y$, where $z = (z_1, ..., z_n)$, $y = (y_1, ..., y_n)$, $x = (x_0, ..., x_s)$, such that

$$z_1^2 + ... + z_n^2 = (x_0^2 + ... + x_s^2)(y_1^2 + ... + y_n^2).$$

This means that $\sum_0^s x_j A_j$ is orthogonal, i.e. leaves invariant $\sum_0^n y_j^2$ except for the factor $\sum_0^s x_j^2$; and thus, since we may assume $A_0 = E$, that $A_1, ..., A_s$ is a set of orthogonal HR-matrices in the sense of (1).

The case U refers to the analogous problem for the identity

$$|z_1|^2 + ... + |z_n|^2 = (x_0^2 + ... + x_s^2)(|y_1|^2 + ... + |y_n|^2)$$

where y and z are complex, and x real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination $\sum_0^s x_j A_j$ of $2n \times 2n$-matrices with $A_0 = E$ is symplectic up to the factor $\sum_0^s x_j^2$ if and only if $A_1, ..., A_s$ is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group $Sp(n) \subset U(2n)$, and write Sp for the infinite symplectic group $\varinjlim Sp(k)$. With a set $A_1, ..., A_s$ of unitary symplectic HR-matrices, and $A_0 = E$, we associate the map $f_s(x_0, x_1, ..., x_s) = \sum_0^s x_j A_j$, $\sum_0^s x_j^2 = 1$, of S^s into Sp via $Sp(n)$; we also write f_s for the corresponding element of $\pi_s(Sp)$, known to be 0 or cyclic.

THEOREM A′. *If $A_1, ..., A_s$ are unitary symplectic HR-matrices of minimal size $2n_0$ then f_s is a generator of $\pi_s(Sp)$.*

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group G_s, $s = 0, 1, 2, ...$ introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the "reduced" Grothendieck groups of representations E_s^U and E_s^O are computed; they turn out to be isomorphic to $\pi_s(U)$ and $\pi_s(O)$ respectively. Moreover a product is defined in the direct sum of the $E_s^U (E_s^O)$ turning it into a graded ring $E_*^U (E_*^O)$. The claim of Theorem A is proved in Section 3; we show that the maps $\phi: E_s^U \to \pi_s(U)$, $\psi: E_s^O \to \pi_s(O)$ given by the f_s of 0.1 are isomorphisms. Using the product structure in $\pi_*(U)$ and $\pi_*(O)$ known from K-theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of U, O and Sp.

1. The groups G_s and their representations

1.1. We will denote throughout by G_s the group given by the presentation

$$G_s = \langle \varepsilon, a_1, ..., a_s \mid \varepsilon^2 = 1, \ a_j^2 = \varepsilon, \ a_j a_k = \varepsilon a_k a_j, \ j, k = 1, 2, ..., s, j \neq k \rangle .$$

Clearly any set $A_1, ..., A_s$ of HR-matrices yields a (unitary or orthogonal) representation of G_s of degree n by $\varepsilon \mapsto -E$, $a_j \mapsto A_j$, $j = 1, 2, ..., s$. Conversely a representation of G_s with $\varepsilon \mapsto -E$, in short an ε-representation, yields a set of s HR-matrices. For the elementary properties of G_s and its representations we refer to [E]. We just recall that the order of G_s is 2^{s+1}, that ε is central, and that the irreducible unitary ε-representations of G_s are of degree $2^{s/2}$ if s is even (one equivalence class), of degree $2^{(s-1)/2}$ if s is odd (two equivalence classes). These degrees are the minimal values n_0 in case U. As for the case O, one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary ε-representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on s yields the minimal values n_0 (case O) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal ε-representations of G_s.

1.2. A very simple and useful scheme for studying the groups G_s and their ε-representations (and many other things) has been deviced by T. Y. Lam and T. Smith [LS]. They have expressed the G_s as products of very small and well-known groups. Namely $C = G_1$, the cyclic group of order 4; $Q = G_2$, the quaternionic group of order 8; K, the Klein 4-group; and D, the dihedral group of order 8. Although K and D do not belong to the family G_s, they are of a similar nature and contain a distinguished central element ε of order 2 (distinguished arbitrarily in K). "Product" here means the central product obtained from the direct product by identifying the