

## §4. An improvement

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$$\begin{aligned} \pi_1(E(L \times I \natural D)) &\simeq \pi_1(E(L \times I)) \underset{\langle m \rangle}{*} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) * (\pi_1(E(D)) / \langle m \rangle) \end{aligned}$$

where the latter isomorphism is because  $\langle m \rangle = 1$  in  $\pi_1(E(L \times I))$  by the assumption. Since  $\pi_1(E(D)) / \langle m \rangle \simeq \pi_1(D^{n+3}) \simeq \{1\}$ , we have

$$(3.8) \quad \pi_1(E(L \times I \natural D)) \simeq \pi_1(E(L \times I)) \simeq \pi_1(E(L)).$$

Here the inclusion map  $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \natural D)$  induces the isomorphism.

We shall observe that  $i$  is a simple homotopy equivalence. For that purpose we consider the lifting of  $i$  to the universal covers. Since the map  $\pi_1(E(D)) \rightarrow \pi_1(E(L \times I \natural D))$  induced by the inclusion map is trivial as observed above, it follows from (3.7) that

$$(3.9) \quad \tilde{E}(L \times I \natural D) = \tilde{E}(L \times I) \cup E(D) \times \Pi$$

where  $\Pi = \pi_1(E(L \times I \natural D)) = \pi_1(M - L)$  and  $\tilde{E}(L \times I)$  and  $E(D) \times \Pi$  are pasted together  $\Pi$ -equivariantly along  $D^{n+1} \times S^1 \times \Pi$  embedded in their boundaries. This means that  $\tilde{i}_*: H_q(\tilde{E}(L); \mathbf{Z}) \rightarrow H_q(\tilde{E}(L \times I \natural D); \mathbf{Z})$  is an isomorphism as  $\mathbf{Z}[\Pi]$ -modules. Hence  $i_*: \pi_q(E(L)) \rightarrow \pi_q(E(L \times I \natural D))$  is an isomorphism by Namioka's theorem (see [W11, §4]) and hence  $i$  is a homotopy equivalence.

The assumption  $\langle m \rangle = 1$  together with (3.9) tells us that the Whitehead torsion  $\tau(i) \in Wh(\Pi)$  of the map  $i$  comes from an element of  $Wh(1)$  through the map:  $Wh(1) \rightarrow Wh(\Pi)$  induced from the inclusion  $1 \rightarrow \Pi$ . However  $Wh(1) = 0$  and hence  $\tau(i) = 0$ . This shows that  $E(L \times I \natural D)$  is an  $s$ -cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where  $n$  is even  $\geq 4$ . It would be interesting to ask if the same conclusion still holds in the case  $n = 2$ .

In the next section we will improve Proposition 3.6 when  $n$  is odd  $\geq 5$ .

#### § 4. AN IMPROVEMENT

Throughout this section we assume  $n$  is odd  $\geq 5$ . Let  $V^{n+1}$  be a Seifert surface of an  $n$ -knot  $K$  in  $S^{n+2}$ . The normal bundle to  $V$  in  $S^{n+2}$  is trivial. We give the stable normal bundle of  $S^{n+2}$  a canonical framing so that  $V$  can be viewed as a framed manifold.

Remember that  $\partial V = K = S^n$ . We make  $V$  contractible by framed surgery without touching the boundary. As is well known this is always possible in case  $\dim V = n + 1$  is odd. But in case  $n + 1$  is even, we encounter an obstruction which is detected by

$$\begin{cases} \text{Sign } V \in \mathbf{Z} & \text{if } n + 1 \equiv 0 \pmod{4} \\ c(V) \in \mathbf{Z}/2\mathbf{Z} & \text{if } n + 1 \equiv 2 \pmod{4} \end{cases} \quad (4)$$

where  $c(V)$  is the Kervaire invariant of  $V$ .

*Remark 4.1.* Since  $\partial V$  is diffeomorphic to  $S^n$ ,  $c(V) = 0$  if  $n + 1$  is not of the form  $2^k - 2$  ([Br]).

One can see that Seifert surfaces of  $K$  are framed cobordant relative boundary to each other. Hence the values  $\text{Sign } V$  and  $c(V)$  are independent of the choice of  $V$ . We set

$$\sigma(S^{n+2}, K) = \begin{cases} \text{Sign } V & \text{if } n + 1 \equiv 0 \pmod{4}, \\ c(V) & \text{if } n + 1 = 2^k - 2 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

**PROPOSITION 4.2.** *Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n$  is odd  $\geq 5$ . Then  $(S^{n+2}, K) \in I_0(M, L)$  if  $\sigma(S^{n+2}, K) = 0$ . In particular,  $I_0(M, L) = \mathcal{H}_n$  if neither  $n + 1 \equiv 0 \pmod{4}$  nor  $n + 1 = 2^k - 2$  for some  $k$ .*

Combining this with Theorem 1.1, we obtain

**COROLLARY 4.3.** *Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n + 1 \equiv 0 \pmod{4}$  ( $n \neq 3$ ). Then  $(S^{n+2}, K) \in I_0(M, L)$  if and only if  $\sigma(S^{n+2}, K) = 0$ .*

The rest of this section is devoted to the proof of Proposition 4.2. Let  $K$  be an  $n$ -knot in  $S^{n+2}$  such that  $\sigma(S^{n+2}, K) = 0$ . We shall construct an  $s$ -cobordism relative boundary between  $E(L \cup K)$  and  $E(L)$ . The argument is rather more complicated than that of Proposition 3.6. We need some knowledge of surgery theory.

*Step 1.* Let  $V^{n+1}$  be a Seifert surface of  $K$ . Push the interior of  $V$  into the interior of  $D^{n+3}$  to make it transverse to the boundary  $S^{n+2}$  of  $D^{n+3}$ . We may assume that  $V$  is  $(n-1)/2$ -connected, if necessary, by doing framed surgery of  $V$  within  $D^{n+3}$ . In fact, this is the method used to prove that any  $n$ -knot is concordant to a simple knot (see [KW, Chap. IV]).

In the attempt to make  $V$   $(n+1)/2$ -connected (and hence  $V$  is contractible by the Poincaré duality) by framed surgery of  $V$  within  $D^{n+3}$ , one encounters an obstruction. Namely a bunch of embedded  $(n+1)/2$ -spheres in  $V$  does

not necessarily extend to embedded  $(n+3)/2$ -disks whose interior lies in  $D^{n+3} - V$ .

But if we do framed surgery of  $V$  at the outside of  $D^{n+3}$  without touching boundary, i.e. if we do surgery on framed embeddings

$$(S^{(n+1)/2} \times D^{(n+1)/2} \times D^2, S^{(n+1)/2} \times D^{(n+1)/2} \times \{0\}) \rightarrow (D^{n+3}, V),$$

then we can make  $V$   $(n+1)/2$ -connected because the obstruction is exactly  $\sigma(S^{n+2}, K)$  and it vanishes by the assumption. The ambient space is, however, not  $D^{n+3}$  any more. We denote by  $(W, D)$  the resulting framed oriented pair, where  $D$  is diffeomorphic to  $D^{n+1}$ .

*Step 2.* We construct a boundary preserving map  $h$ :

$$(W; N(D), E(D)) \rightarrow (D^{n+3}; N(D^{n+1}), E(D^{n+1}))$$

such that

$$(4.4) \quad h|_{\partial W}: \partial W = S^{n+2} \rightarrow \partial D^{n+3} = S^{n+2} \quad \text{is a homotopy equivalence,}$$

$$(4.5) \quad h|_{N(D)}: N(D) \rightarrow N(D^{n+1}) \quad \text{is a diffeomorphism,}$$

where  $N$  denotes a closed tubular neighborhood and  $D^{n+1} \subset D^{n+3}$  is standardly embedded.

Since  $D$  is diffeomorphic to  $D^{n+1}$ , there is a diffeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times \{0\}) \rightarrow (N(D), D).$$

Here  $D^{n+1} \times D^2$  can be naturally identified with  $N(D^{n+1})$ ; so we define

$$(4.6) \quad h|_{N(D)} = g^{-1}$$

First we extend  $h|_{\partial W \cap \partial N(D)} = h|_{\partial E(K)}$  to a map from  $E(K)$  to  $E(\partial D^{n+1}) = E(S^n)$ . The obstruction lies in groups

$$H^{q+1}(E(K), \partial E(K); \pi_q(E(S^n))).$$

Since  $E(S^n)$  is homotopy equivalent to  $S^1$ , it suffices to prove

$$(4.7) \quad H^{q+1}(E(K), \partial E(K); \mathbf{Z}) = 0 \quad \text{for } q = 0, 1.$$

On the other hand we have

$$\begin{aligned} H^{q+1}(E(K), \partial E(K); \mathbf{Z}) &\simeq H^{q+1}(S^{n+2}, N(K); \mathbf{Z}) && \text{(by excision)} \\ &\simeq \tilde{H}^q(N(K); \mathbf{Z}) && \text{(if } q+1 < n+2) \\ &\simeq \tilde{H}^q(S^n; \mathbf{Z}) \\ &= 0 && \text{(if } q \neq n) \end{aligned}$$

Hence (4.7) is satisfied as  $n \geq 5$ .

Consequently we can extend  $h|_{N(D)}$  to a map

$$h|_{N(D) \cup \partial W}: (N(D) \cup \partial W, \partial W) \rightarrow (N(D^{n+1}) \cup \partial D^{n+3}, \partial D^{n+3}).$$

The local degree of  $h|_{\partial W}: \partial W \rightarrow \partial D^{n+3}$  is one because  $h|_{\partial W \cap N(D)} = h|_{N(K)}: N(K) \rightarrow N(S^n)$  is a diffeomorphism by (4.6) and  $h(E(K)) \subset E(S^n)$  by the construction. Since  $\partial W$  and  $\partial D^{n+3}$  are both  $S^{n+2}$ ,  $h|_{\partial W}$  is a homotopy equivalence. Hence (4.4) is satisfied. Moreover (4.5) is also satisfied by (4.6). In the sequel it suffices to extend  $h|_{\partial E(D)}$  to a map from  $E(D)$  to  $E(D^{n+1})$ . This time the obstruction lies in groups

$$H^{q+1}(E(D), \partial E(D); \pi_q(E(D^{n+1}))).$$

Since  $E(D^{n+1})$  is homotopy equivalent to  $S^1$ , it suffices to prove

$$(4.8) \quad H^{q+1}(E(D), \partial E(D); \mathbf{Z}) = 0 \quad \text{for } q = 0, 1.$$

By excision we have

$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) \simeq H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}).$$

Remember that  $W$  is obtained from  $D^{n+3}$  by  $(n+1)/2$ -surgery. It implies that

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{if } i \neq (n+1)/2 + 1.$$

In particular

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{for } i \leq 3$$

as  $n \geq 5$ . Therefore it follows from the exact sequence of the pair  $(W, N(D) \cup \partial W)$  that

$$H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}) \simeq \tilde{H}^q(N(D) \cup \partial W; \mathbf{Z}) \quad \text{for } q \leq 2.$$

Here the Mayer-Vietoris exact sequence of the triad  $(N(D) \cup \partial W; N(D), \partial W)$  shows that

$$\tilde{H}^q(N(D) \cup \partial W; \mathbf{Z}) = 0 \quad \text{for } q = 0, 1,$$

because  $N(D)$  is contractible,  $\partial W = S^{n+2}$ , and  $N(D) \cap \partial W = S^n \times S^1$ . Hence (4.8) is satisfied, and we have obtained the desired map  $h$ .

*Step 3.* Since  $W$  is framed, the framing of the stable normal bundle  $\nu(W)$  of  $W$  induces a stable bundle map  $b: \nu(W) \rightarrow \nu(D^{n+3})$  which covers  $h$ . The triple  $\mathcal{B} = (W, h, b)$  is called a normal map.

The identity map  $Id: (M, L) \times I \rightarrow (M, L) \times I$  gives a normal map where the stable bundle map is also the identity. We shall denote the normal

map by  $\mathcal{B}_{Id} = ((M, L) \times I, Id, Id)$ . The maps  $h$  and  $Id$  are both diffeomorphisms on  $N(D)$  and  $N(L \times I)$  respectively; so one can do the boundary connected sum of  $\mathcal{B}$  and  $\mathcal{B}_{Id}$  at points of  $K$  and  $L \times \{1\}$ . This yields a new normal map  $\mathcal{B}_{Id} \sharp \mathcal{B} = (M \times I \sharp W, Id \sharp h, Id \sharp b)$ . Here we naturally identify the target space  $(M, L) \times I \sharp (D^{n+3}, D^{n+1})$  with  $(M, L) \times I$ . Since  $Id \sharp h$  is a diffeomorphism on  $N(L \times I \sharp D)$ , it gives a product structure on  $N(L \times I \sharp D)$ . Thus we get a cobordism  $E(L \times I \sharp D)$  relative boundary between  $E(L \sharp K)$  and  $E(L)$ .

*Step 4.*  $Id \sharp h|_{E(L)}: E(L) \rightarrow E(L) \times \{0\}$  (the 0-level) is the identity; so it is a simple homotopy equivalence. We shall observe that  $h_1 = Id \sharp h|_{E(L \sharp K)}: E(L \sharp K) \rightarrow E(L) \times \{1\}$  (the 1-level) is also a simple homotopy equivalence.

We have a decomposition

$$E(L \sharp K) = E(L) \cup E(K)$$

in the same sense as (3.7). Hence, similarly to (3.8) one can see

$$(4.9) \quad \pi_1(E(L \sharp K)) \simeq \pi_1(E(L))$$

where the inclusion map induces the isomorphism.

We can view  $E(L) \times \{1\}$  as  $E(L \sharp S^n)$  and we also have

$$E(L \sharp S^n) = E(L) \cup E(S^n).$$

Then the map  $h_1$  can be viewed as the identity on  $E(L)$  and  $h$  on  $E(K)$ . This together with (4.9) shows that  $h_{1*}: \pi_1(E(L \sharp K)) \rightarrow \pi_1(E(L \sharp S^n))$  is an isomorphism.

As before we consider the map  $\tilde{h}_1: \tilde{E}(L \sharp K) \rightarrow \tilde{E}(L \sharp S^n)$  lifted to the universal covers. Since  $\langle m \rangle = 1$ , we have a diagram

$$(4.10) \quad \begin{array}{ccc} \tilde{E}(L \sharp K) & = & \tilde{E}(L) \cup E(K) \times \Pi \\ \tilde{h}_1 \downarrow & & \downarrow Id \quad \downarrow h|_{E(K)} \times Id \\ \tilde{E}(L \sharp S^n) & = & \tilde{E}(L) \cup E(S^n) \times \Pi, \end{array}$$

where  $\Pi = \pi_1(M - L)$  as before. Since  $h|_{E(K)}$  is a homology equivalence, the above diagram tells us that  $\tilde{h}_{1*}: H_q(\tilde{E}(L \sharp K); \mathbf{Z}) \rightarrow H_q(\tilde{E}(L \sharp S^n); \mathbf{Z})$  is an isomorphism as  $\mathbf{Z}[\Pi]$ -modules. Therefore  $h_1$  is a homotopy equivalence by the same reason as before.

The assumption  $\langle m \rangle = 1$  together with the above diagram tells us that  $\tau(h_1) \in Wh(\Pi)$  comes from an element of  $Wh(1)$ . Hence  $\tau(h_1) = 0$  as  $Wh(1) = 0$ .

Step 5. By step 4  $\bar{h} = Id \natural h|_{E(L \times I \natural D)}: E(L \times I \natural D) \rightarrow E(L \times I \natural D^{n+1}) = E(L \times I)$  is a simple homotopy equivalence on the boundary. We convert  $\bar{h}$  into a simple homotopy equivalence by surgery without touching the boundary. The obstruction  $\sigma(\bar{h})$  lies in an  $L$ -group  $L_{n+3}(\Pi, 1)$  where 1 denotes the trivial homomorphism from  $\Pi$  to  $\mathbf{Z}_2$  (note, since  $M$  is oriented and hence so is  $E(L \times I)$ , the orientation homomorphism:  $\Pi = \pi_1(E(L \times I)) \rightarrow \mathbf{Z}_2$  is trivial).

We have a diagram similar to (4.10):

$$\begin{array}{ccc} E(L \times I \natural D) & = & E(L \times I) \cup E(D) \\ \bar{h} \downarrow & & \downarrow Id \quad \downarrow h \\ E(L \times I \natural D^{n+1}) & = & E(L \times I) \cup E(D^{n+1}). \end{array}$$

The surgery obstruction  $\sigma(h)$  to converting  $h$  to a simple homotopy equivalence by surgery without touching the boundary lies in  $L_{n+3}(\mathbf{Z}, 1)$  because  $\pi_1(E(D^{n+1}))$  is isomorphic to  $\mathbf{Z}$ . The above diagram together with the assumption  $\langle m \rangle = 1$  tells us that

$$\sigma(\bar{h}) = \beta_* \alpha_* \sigma(h)$$

where  $\alpha_*: L_{n+3}(\mathbf{Z}, 1) \rightarrow L_{n+3}(1, 1)$  and  $\beta_*: L_{n+3}(1, 1) \rightarrow L_{n+3}(\Pi, 1)$  are the homomorphisms induced from the trivial homomorphisms  $\alpha: \mathbf{Z} \rightarrow 1$  and  $\beta: 1 \rightarrow \Pi$  respectively. It is well-known that

$$L_{n+3}(1, 1) \simeq \begin{cases} \mathbf{Z} & \text{if } n+3 \equiv 0 \pmod{4}, \\ \mathbf{Z}_2 & \text{if } n+3 \equiv 2 \pmod{4}. \end{cases}$$

As easily observed  $\alpha_* \sigma(h)$  is given by

$$\begin{cases} \text{Sign } W & \text{if } n+3 \equiv 0 \pmod{4} \\ c(W) & \text{if } n+3 \equiv 2 \pmod{4} \end{cases}$$

through the above isomorphism. Remember that  $W$  is framed cobordant to  $D^{n+3}$  relative boundary by the construction. Therefore those invariants vanish and hence  $\sigma(\bar{h}) = 0$ .

Consequently we have obtained a cobordism  $U'$  relative boundary between  $E(L \natural K)$  and  $E(L)$  together with a simple homotopy equivalence  $F: U' \rightarrow E(L \times I)$  which is the identity on the 0-level. Let  $i_0: E(L) \rightarrow U'$  and  $j_0: E(L) \rightarrow E(L \times I)$  be the inclusion maps from the 0-level to the cobordisms. Since  $F \circ i_0 = j_0 \circ Id$  where  $Id: E(L) \rightarrow E(L)$  denotes the identity map, we have

$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [M1, Lemma 7.8]). Here  $F$ ,  $j_0$ , and  $Id$  are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that  $\tau(i_0) = 0$ , because  $F_*: Wh(\pi_1(U')) \rightarrow Wh(\pi_1(E(L \times I)))$  is an isomorphism. This means that  $U'$  is an  $s$ -cobordism. Therefore  $(S^{n+2}, K) \in I_0(M, L)$  by Lemma 1.6. Q.E.D.

### § 5. TYPE 3 CASE

In this section we treat the case where  $\langle m \rangle$  or  $[m]$  is of order  $p$  ( $p$  is not necessarily a prime number). We begin with

LEMMA 5.1. *Suppose  $[m]$  is of order  $p$ . Then if  $(S^{n+2}, K) \in I(M, L)$ , then  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere.*

*Proof.* Let  $r$  be the order of  $\text{Tor } H_1(M-L; \mathbf{Z})$ , and let  $\gamma$  be the canonical epimorphism  $\pi_1(M-L) \rightarrow H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$ . Since the order of  $\gamma(\langle m \rangle)$  is  $p$ , we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If  $p \geq 2$ , there are infinitely many knots  $(S^{n+2}, K)$  such that  $(S^{n+2}, K)_p$  is not a homotopy  $(n+2)$ -sphere; so Lemma 5.1 shows that  $I(M, L) \subsetneq \mathcal{K}_n$  for such  $(M, L)$ .

The rest of this section is devoted to looking for a non-trivial knot in  $I(M, L)$  or  $I_0(M, L)$ . We will extend Proposition 3.6 and 4.2 to the case where  $\langle m \rangle$  is of order  $p$ . Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let  $(S^{n+2}, K)$  be an  $n$ -knot which bounds a disk pair  $(D^{n+3}, D)$  such that  $(D^{n+3}, D)_p$  is a homotopy  $(n+3)$ -disk. Since  $(S^{n+2}, K)_p$  is the boundary of  $(D^{n+3}, D)_p$ ,  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere. If  $n+3 \geq 5$ , then  $(D^{n+3}, D)_p$  is diffeomorphic to  $D^{n+3}$  and hence  $(S^{n+2}, K)_p$  is diffeomorphic to  $S^{n+2}$ .

The  $p$ -fold branched cyclic covering  $(D^{n+3}, D)_p$  supports a  $\mathbf{Z}_p$ -action with the branch set  $D$  as the fixed point set. Let  $E(D)_p$  be the exterior of  $D$  in  $(D^{n+3}, D)_p$  and let  $\rho: S^1 \rightarrow E(D)_p$  be an equivariant embedding of a meridian of  $D$  in  $E(D)_p$ , where the standard free  $\mathbf{Z}_p$ -action is considered on  $S^1$ . Since  $\rho$  is a homology equivalence and equivariant, the Whitehead torsion of  $\rho$  is defined in  $Wh(\mathbf{Z}_p)$ . Clearly it is independent of the choice of  $\rho$ ; so we shall denote it by  $\tau_p(D^{n+3}, D)$ .

The following theorem is an extension of Proposition 3.6.