ON CHERN CLASSES OF FINITE GROUP REPRESENTATIONS

Autor(en): **Kroll, Ove**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-57380

L'Enseignement Mathématique, t. 35 (1989), p. 363-374

ON CHERN CLASSES OF FINITE GROUP REPRESENTATIONS

by Ove KROLL

In his paper "Characters and cohomology of finite groups", Atiyah observed that for a finite group G and a complex representation

$$\rho: G \to Gl_n(\mathbf{C})$$

of dimension n, it is possible to attach to ρ the natural cohomology classes with integral coefficients

$$c_i(\rho) \in H^{2i}(G, \mathbf{Z}), \quad i = 0, 1, ..., n,$$

namely the Chern classes of a vector bundle constructed from ρ over the classifying space BG of G.

As the objects involved, group cohomology and complex representations of finite groups, are both algebraic objects (the latter by Brauer's famous theorem stating that complex representations can be realized over finite cyclotomic extensions of the rational numbers), a purely algebraic construction ought to exist.

My aim in this paper is to provide such an algebraic definition of the Chern classes. Another algebraic though quite different construction has been given by Grothendieck in [5], while Evens in [2] has given an algebraic "in principle" way of calculating the classes. In this connection it is also worth mentioning that a formula for the Chern classes of an induced representation from H to G, where H is normal in G of index a prime p, can be found in [3]. However, as already shown in [7], such a formula is not needed to characterize Chern classes of finite group representations.

My construction will be based on the purely algebraic construction of $H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$: the cohomology of the general linear group over the algebraic closure $\bar{k}_p$ of the field with p elements with coefficients in the l-adic integers $\hat{\mathbf{Z}}_l$ (l a prime different from p), which I gave in [6].

Let me state this theorem:

THEOREM 1. *$H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(\sigma_1, ..., \sigma_n)$, where P denotes a polynomial algebra defined over $\hat{\mathbf{Z}}_l$. Furthermore σ_i has degree $2i$.*

Let me also recall that for the subgroup $T_n(\bar{k}_p)$ of diagonal matrices

$$H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, ..., x_n),$$

where x_i has degree 2. Finally, under the restriction map res

$$\operatorname{res}(\sigma_i) = x_1 x_2 \dots x_i + \dots + x_{n-i+1} x_{n-i+2} \dots x_n$$

(the i'th elementary symmetric polynomial).

This theorem clearly allows a definition of p-modular l-adic Chern classes: Given any group G and any p-modular representation defined over $\bar{k}_p$

$$\rho : G \to Gl_n(\bar{k}_p),$$

define $c_i(\rho) \in H^{2i}(G, \hat{\mathbf{Z}}_l)$ by $c_i(\rho) = \rho^*(\sigma_i)$. These classes have the following properties:

THEOREM 2.

CH1. If $f: H \to G$ is a group homomorphism, then

$$c_i(\rho \circ f) = f^*(c_i(\rho)).$$

CH2. Let

$$0 \to \rho_1 \to \rho \to \rho_2 \to 0$$

be an exact sequence of representations (or more precisely of the corresponding modules). If

$$c.(\rho) = 1 + c_1(\rho) + \dots + c_n(\rho)$$

is the total Chern class, then

$$c.(\rho) = c.(\rho_1) c.(\rho_2)$$

CH3. Let G be a locally finite group. Then the product of the c_1's taken over all primes l different from p defines an isomorphism between one-dimensional representations over $\bar{k}_p$ and

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong H^2(G, \prod_{l \neq p} \hat{\mathbf{Z}}_l).$$

Let $R_p(G)$ be the Grothendieck group of representations of G over $\bar{k}_p$ (so $R_p(G)$ is the free abelian group with basis the isomorphism classes of simple $\bar{k}_p G$-modules) and let

$$H^{**}(G, \hat{\mathbf{Z}}_l) = \prod_{n=0}^{\infty} H^{2n}(G, \hat{\mathbf{Z}}_l).$$

As the total Chern class $c.(\rho)$ of a representation ρ is invertible in $H^{**}(G, \hat{\mathbf{Z}}_l)$ (use the formula $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ where x has strictly positive degree), CH2 allows a unique extension of $c.$ to

$$c.: R_p(G) \to H^{**}(G, \hat{\mathbf{Z}}_l).$$

As shown in CH3, Chern classes are closely connected to the cohomology of k_p^* ($k_p^* = Gl_1(\bar{k}_p)$).

PROPOSITION 3. *Let* μ_∞ *be the infinite group of complex roots of unity and define*

$$\hat{\mathbf{Z}} = \prod_l \hat{\mathbf{Z}}_l,$$

the product taken over all primes l. *Then*

i) $H^*(\mu_\infty, \mathbf{Z}) \cong \hat{\mathbf{Z}}[u]$,
where the right hand side denotes a polynomial algebra over $\hat{\mathbf{Z}}$ *in one variable* u *of degree 2 (except that* $H^0(\mu_\infty, \mathbf{Z}) = \mathbf{Z}$*).*

ii) $H^*(\mu_\infty, \hat{\mathbf{Z}}_l) = \hat{\mathbf{Z}}_l[u]$.

iii) $H^*(k_p^*, \hat{\mathbf{Z}}_l) = \hat{\mathbf{Z}}_l[u_p]$
where l *is a prime different from* p *and* u_p *has degree 2.*

iv) $H^n(k_p^*, \prod_{l \neq p} \hat{\mathbf{Z}}_l) \cong H^n(k_p^*, \mathbf{Z})$ *for* $n > 0$.

Now, as $Gl_n(\bar{k}_p)$, $n = 1, 2, 3, \ldots$, are locally finite groups, it follows quite easily from CH1, CH2 and CH3 that p-modular Chern classes with $\hat{\mathbf{Z}}_l$-coefficients are in one-to-one correspondence with $\hat{\mathbf{Z}}_l$ generators u_p of $H^2(k_p^*, \hat{\mathbf{Z}}_l)$. Using this correspondence, I shall say that a system of p-modular l-adic Chern classes is defined by the element u_p in $H^2(k_p^*, \hat{\mathbf{Z}}_l)$.

To define Chern classes of complex representations, I will use the well-known decomposition map d_p from modular representation theory (see e.g. [9]). So let $R_{\mathbf{C}}(G)$ be the Grothendieck group (or character ring) of complex representations of G. Making a choice of a multiplicative embedding

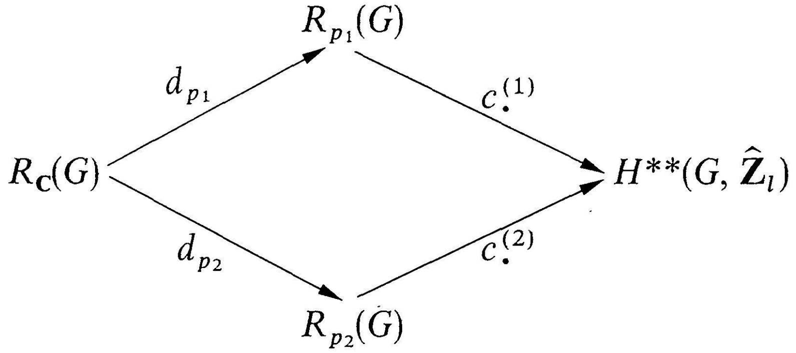
$$e_p: k_p^* \hookrightarrow \mu_\infty,$$

d_p can be defined as follows: If χ is an ordinary character of degree n, $d_p(\chi)$ is a modular representation whose composition factors are uniquely determined by the modular character

$$d_p(\chi)(g) = e_p^{-1}(\lambda_1 + \ldots + \lambda_n),$$

where g is a p-regular element (i.e. of order prime to p) with eigenvalues $\lambda_1, \ldots, \lambda_n$ on the representation determined by χ.

THEOREM 4. *Let $u \in H^2(\mu_\infty, \hat{\mathbf{Z}}_l)$ be a generator, and let p_1 and p_2 be primes, both different from l, and let $e_{p_i}^*(u) \in H^2(k_{p_i}^*, \hat{\mathbf{Z}}_l)$ define p_i-modular Chern classes $c_\cdot^{(i)}$. Then the diagram*



is commutative for all finite groups G.

Now for a finite group G, the unique ring homomorphism

$$\mathbf{Z} \to \prod_l \hat{\mathbf{Z}}_l = \hat{\mathbf{Z}}$$

induces an isomorphism (as the quotient $\hat{\mathbf{Z}}/\mathbf{Z}$ is divisible)

$$H^*(G, \mathbf{Z}) \cong H^*(G, \prod_l \hat{\mathbf{Z}}_l) \cong \prod_l H^*(G, \hat{\mathbf{Z}}_l),$$

and I can now (independently of p by Theorem 4) define the l-primary component of the Chern classes $c_\cdot$ defined by $u \in H^2(\mu_\infty, \hat{\mathbf{Z}})$ of

$$\rho : G \to Gl_n(\mathbf{C})$$

by

$$(c_\cdot(\rho))_l = c_\cdot(d_p(\rho)),$$

where p is a prime different from l and the $c_\cdot$ on the right hand side is the p-modular Chern classes defined by $e_p^*(u_l)$, where u_l is the l-primary component of $u \in H^2(\mu_\infty, \hat{\mathbf{Z}}) = \prod_l H^2(\mu_\infty, \hat{\mathbf{Z}}_l)$.

THEOREM 5. *Let $c_\cdot$ denote the integral Chern classes defined by $u \in H^2(\mu_\infty, \hat{\mathbf{Z}})$. Then*

CH1. Let $f : H \to G$ be a group homomorphism. Then

$$c(\rho\circ f) = f^*(c.(\rho)).$$

CH2. $c.(\rho_1 \oplus \rho_2) = c.(\rho_1)\cdot c.(\rho_2)$.

CH3. $c_1 : \mathrm{Hom}(G, \mathbf{C}^*) \to H^2(G, \mathbf{Z})$ *is an isomorphism and can be described as follows: For* $\varphi \in \mathrm{Hom}(G, \mathbf{C}^*)$, *let* φ *also denote its unique factorization*

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & C^* \\ & \searrow^{\varphi} & \cup \\ & & \mu_\infty \end{array}$$

Now $c_1(\varphi) = \varphi^*(u)$.

Remark. As shown in [7], CH1, CH2 and CH3 uniquely determine the Chern classes defined by u. As different choices of u clearly defines different Chern classes (just observe that

$$H^2(\mu_\infty, \mathbf{Z}) \cong \varinjlim H^2(G_i, \mathbf{Z}),$$

the limit taken over all finite cyclic subgroups), there is a one-to-one correspondence between Chern classes on finite groups and $\hat{\mathbf{Z}}$ generators of $H^2(\mu_\infty, \mathbf{Z})$.

This paper has been organized as follows.

Theorem 2 is proved in Section 1, Theorem 4 in Section 2, and Theorem 5 in Section 3. Proposition 3 i) was proved in [7], and the remaining part of this proposition can be obtained similarly.

Finally, in Section 4 it is shown that there exists a very simple extension of the theory of Chern classes on finite groups to locally finite groups.

I would like to thank Jørgen Tornehave for a helpful conversation.

Section 1. Proof of Theorem 2

CH1 is quite trivial, so let me first prove CH2. Let $\dim \rho_i = n_i$, $\dim \rho = n$, so that $n_1 + n_2 = n$. By assumption, ρ factors through the parabolic subgroup $P = P(\bar{k}_p)$

$$P = \left\{ \begin{matrix} \overset{n_1}{*} & \overset{n_2}{*} \\ 0 & * \end{matrix} \right\}$$

which is isomorphic to a semi-direct product of $Gl_{n_1}(\bar{k}_p) \times Gl_{n_2}(\bar{k}_p)$ acting on a unipotent subgroup U.

As U is a direct limit of p-groups,

$$H^k(U, \hat{\mathbf{Z}}_l) = 0 \quad \text{for } k > 0 .$$

Thus

$$\begin{aligned} H^*(P, \hat{\mathbf{Z}}_l) &\cong H^*(Gl_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \otimes H^*(Gl_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \\ &\cong P(\alpha_1, ..., \alpha_{n_1}) \otimes P(\beta_1, ..., \beta_{n_2}) . \end{aligned}$$

Let

$$H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(\sigma_1, ..., \sigma_n)$$

and

$$H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, ..., x_n) .$$

As $T_n(\bar{k}_p) \cong T_{n_1}(\bar{k}_p) \times T_{n_2}(\bar{k}_p)$, I shall consider

$$H^*(T_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, ..., x_{n_1})$$

and

$$H^*(T_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_{n_1+1}, ..., x_n)$$

as contained in $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Furthermore, as all restriction maps are injective, I shall view $H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$ and $H^*(Gl_{n_i}(\bar{k}_p), \hat{\mathbf{Z}}_l)$, $i = 1, 2$, as subspaces of $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Thus

$$\begin{aligned} \alpha_i &= \text{the } i\text{'th elementary symmetric polynomial in } x_1, ..., x_{n_1} \\ \beta_i &= \text{the } i\text{'th elementary symmetric polynomial in } x_{n_1+1}, ..., x_n \\ \sigma_i &= \text{the } i\text{'th elementary symmetric polynomial in } x_1, ..., x_n . \end{aligned}$$

Furthermore, the formula

$$c.(\rho_1 \oplus \rho_2) = c.(\rho_1 \oplus \rho_2)$$

is equivalent to

$$1 + \sigma_1 t + ... + \sigma_n t^n = (1+\alpha_1 t+...+\alpha_{n_1} t^{n_1}) \cdot (1+\beta_1 t+...+\beta_{n_2} t^{n_2}) ,$$

and this follows from the identity

$$\begin{aligned} \sum_{i=0}^{n} \sigma_i t^i &= \prod_{i=1}^{n} (1+tx_i) = \prod_{i=1}^{n_1} (1+tx_i) \cdot \prod_{i=n_1+1}^{n_2} (1+tx_i) \\ &= \left(\sum_{i=0}^{n_1} \alpha_i t^i\right)\left(\sum_{i=0}^{n_2} \beta_i \cdot t^i\right) . \end{aligned}$$

To prove CH3, observe that for G locally finite the homology groups $H_i(G, \mathbf{Z})$ are all torsion groups for $i > 0$ as

$$H_i(G, \mathbf{Z}) \cong \varinjlim H_i(G_k, \mathbf{Z}),$$

the limit taken over a family of finite subgroups G_k of G such that $\varinjlim G_k = G$. Now, by the universal coefficient theorem,

$$0 \to \operatorname{Ext}^1_{\mathbf{Z}}(H_1(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \to H^2(G, \hat{\mathbf{Q}}_l) \to \operatorname{Hom}_{\mathbf{Z}}(H_2(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \to 0$$

is exact ($\hat{\mathbf{Q}}_l$ is the quotient field of $\hat{\mathbf{Z}}_l$) so it follows that $H^2(G, \hat{\mathbf{Q}}_l) = 0$ as $\hat{\mathbf{Q}}_l$ is both torsion-free and divisible. From the long exact sequence in cohomology it now follows that

$$H^1(G, \hat{\mathbf{Q}}_l/\hat{\mathbf{Z}}_l) \cong H^2(G, \hat{\mathbf{Z}}_l).$$

Finally, as $\hat{\mathbf{Q}}_l/\mathbf{Z}_l \cong C_{l^\infty}$, where C_{l^∞} is the injective hull of a cyclic l-group, it follows that

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong \prod_{l \neq p} H^1(G, C_{l^\infty}) \cong H^1(G, \prod_{l \neq p} C_{l^\infty}) = H^1(G, \bigoplus_{l \neq p} C_{l^\infty}).$$

The last equality holds, as G is locally finite and $\bigoplus_{l \neq p} C_{l^\infty}$ is the torsion subgroup of $\prod_{l \neq p} C_{l^\infty}$.

Section 2. Proof of Theorem 4

Let G be a given finite group of order $|G|$ and

$$\rho: G \to Gl_n(\mathbf{C})$$

a complex representation.

Choose q to be a power of a prime number p different from l such that

$$q \equiv 1 \text{ (modulo } |G|)$$

Define

$$\phi: Gl_n(q) \to \mathbf{C}$$

by

$$\phi(g) = \sum_{i=1}^{n} e_p(\lambda_i)$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of g. As shown by J. A. Green in [4], ϕ is a virtual complex character of $Gl_n(q)$.

Furthermore let

$$f : G \to Gl_n(q)$$

be the mod-p reduction of ρ to $Gl_n(q)$. (It factors through $Gl_n(q)$, as all $|G|$-roots of unity are contained in the Galois field $GF(q)$ with q elements).

Let $f^* : R_{\mathbf{C}}(Gl_n(q)) \to R_{\mathbf{C}}(G)$ be a map induced on complex character rings by f. By inspection

$$f^*(\phi) = \rho .$$

Let $a = \nu_l(q-1)$, where ν_l is the l-adic valuation and let

$$p^{**} : H^{**}(G, \hat{\mathbf{Z}}_l) \to H^{**}(G, \mathbf{Z}_{l^a})$$

be the map induced by the projection $p : \hat{\mathbf{Z}}_l \to \mathbf{Z}_{l^a}$. Clearly p^{**} is injective in positive dimensions, as multiplication by l^a is zero on $H^{**}(G, \hat{\mathbf{Z}}_l)$.

Now the following diagram is commutative

$$\begin{array}{ccc}
H^{**}(G, \hat{\mathbf{Z}}_l) & \overset{p^{**}}{\rightarrow} & H^{**}(G, \mathbf{Z}_{l^a}) \\
f^{**} \uparrow & & \uparrow f^{**} \\
H^{**}(Gl_n(q), \hat{\mathbf{Z}}_l) & \rightarrow & H^{**}(Gl_n(q), \mathbf{Z}_{l^a}) \\
\downarrow \text{res} & & \downarrow \text{res} \\
H^{**}(T_n(q), \hat{\mathbf{Z}}_l) & \overset{p^{**}}{\rightarrow} & H^{**}(T_n(q), \mathbf{Z}_{l^a})
\end{array}$$

where the restriction map on the right is injective as shown in [6].

Thus for $i = 1, 2$

$$c_{\bullet}^{(i)}(d_{p_i}(\rho)) = (p^{**})^{-1} f^{**}(\text{res})^{-1} p^{**}(d_{p_i}(\tau))$$

where τ is the restriction of the virtual character ϕ to $T_n(q)$. [Note that $(p^{**})^{-1}$ and $(\text{res})^{-1}$ both make sense as the above diagram is commutative].

Thus to show equality, it suffices using CH1 in Theorem 2, to show that

$$c_{\bullet}^{(1)}(d_{p_1}(\tau)) = c_{\bullet}^{(2)}(d_{p_2}(\tau))$$

But $T_n(q)$ is abelian so τ is a direct sum of n one-dimensional representations. By CH2 of Theorem 2 it suffices to show that for a one dimensional representation

$$\varphi\colon T_n(q) \to \mu_\infty,$$
$$c_\bullet^{(1)}\big(d_{p_1}(\varphi)\big) = c_\bullet^{(2)}\big(d_{p_2}(\varphi)\big).$$

But

$$d_{p_i}(\varphi) = e_{p_i}^{-1} \circ \varphi$$

and

$$c_\bullet^{(i)}\big(d_{p_i}(\varphi)\big) = \varphi^* \circ (e_{p_i}^{-1})^* \big(e_{p_i}^*(u)\big) = \varphi^*(u).$$

Remark. It is necessary to reduce to $\mathbf{Z}_{l^a}$ coefficients as the restriction map

$$H^*\big(Gl_n(q), \hat{\mathbf{Z}}_l\big) \to H^*\big(T_n(q), \hat{\mathbf{Z}}_l\big)$$

is not injective in general.

SECTION 3. PROOF OF THEOREM 5

CH1 and CH2 clearly follow from resp. CH1 and CH2 in Theorem 2 together with the functoriality of the decomposition map d_p i.e. the diagram

$$\begin{array}{ccc} R_{\mathbf{C}}(G) & \xrightarrow{f^*} & R_{\mathbf{C}}(H) \\ \downarrow d_p & & \downarrow d_p \\ R_p(G) & \xrightarrow{f^*} & R_p(H) \end{array}$$

is commutative for a group homomorphism $f\colon H \to G$. To obtain CH3 note that $d_p(\varphi) = e_p^{-1} \circ \varphi$ so by definition

$$c_1(\varphi) = c_1\big(d_p(\varphi)\big) = (e_p^{-1}\circ\varphi)(e_p^* u) = \varphi^* \circ (e_p^{-1})^* \circ e_p^*(u) = \varphi^*(u).$$

Furthermore let δ be the connecting homomorphism obtained from the exact sequence

$$\mathbf{Z} \rightarrowtail \mathbf{Q} \twoheadrightarrow \mathbf{Q}/\mathbf{Z}$$

As the diagram

$$\begin{array}{ccc} H^1(\mu_\infty, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\sim\delta} & H^2(\mu_\infty, \mathbf{Z}) \\ \downarrow \varphi^* & & \downarrow \varphi^* \\ H^1(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\sim\delta} & H^2(G, \mathbf{Z}) \end{array}$$

is commutative and as both δ's are isomorphisms it suffices to show that

$$\delta^{-1} \circ c_1 : \operatorname{Hom}(G, \mathbf{C}^*) \to H^1(G, \mathbf{Q}/\mathbf{Z}) \simeq \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$$

is an isomorphism.

But by inspection $\delta^{-1} \circ c_1(\varphi) = \delta^{-1}(u) \circ \varphi$, and as u is a $\hat{\mathbf{Z}} = \operatorname{End}_{\mathbf{Z}}(\mu_\infty)$ generator for $H^2(\mu_\infty, \mathbf{Z}) \simeq \hat{\mathbf{Z}}$, $\delta^{-1}(u)$ is an isomorphism

$$\delta^{-1}(u) : \mu_\infty \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}\,.$$

SECTION 4. CHERN CLASSES FOR LOCALLY FINITE GROUPS

The definition will be based on the following two observations. In the following, let $G = \varinjlim G_k$ be a locally finite group where $\{G_k\}$ is a family of finite subgroups.

LEMMA. *Let*

$$\varphi : G \to Gl_n(\mathbf{C})$$

be a representation of G. *Then* φ *is uniquely determined by its restrictions*

$$\varphi_k : G_k \to Gl_n(\mathbf{C})\,.$$

Conversely given a family of compatible representations $\varphi_k : G_k \to Gl_n(\mathbf{C})$, *there exists a unique* $\varphi : G \to Gl_n(\mathbf{C})$ *which restricts to* φ_k *for all* k.

Proof. From the universal property of the direct limit, we have

$$\operatorname{Hom}(G, Gl_n(\mathbf{C})) \cong \varprojlim \operatorname{Hom}(G_k, Gl_n(\mathbf{C}))\,.$$

PROPOSITION. *For all* $i \geqslant 0$, *the natural map*

$$H^i(G, \mathbf{Z}) \cong \varprojlim H^i(G_k, \mathbf{Z})\,.$$

is an isomorphism.

Proof. Obvious for $i = 0, 1$. For $i \geqslant 1$, the homology groups

$$H_i(G, \mathbf{Z}) = \varinjlim H_i(G_k, \mathbf{Z})$$

are all abelian torsion groups.

Now by the universal coefficient theorem $(i \geqslant 1)$

$$H^i(G, \mathbf{Z}) \cong \operatorname{Ext}^1_{\mathbf{Z}}(H_{i-1}(G, \mathbf{Z}), \mathbf{Z})$$

and as observed above, for $i \geqslant 2$, $H_{i-1}(G, \mathbf{Z})$ is torsion. Thus

$$\begin{aligned} \operatorname{Ext}^1_{\mathbf{Z}}(H_{i-1}(G, \mathbf{Z}), \mathbf{Z}) &\cong \operatorname{Hom}_{\mathbf{Z}}(H_{i-1}(G, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \\ \cong \operatorname{Hom}_{\mathbf{Z}}(\varinjlim H_{i-1}(G_k, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) &\cong \varprojlim \operatorname{Hom}_{\mathbf{Z}}(H_{i-1}(G_k, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \\ &\simeq \varprojlim \operatorname{Ext}^1_{\mathbf{Z}}(H_{i-1}(G_k, \mathbf{Z}), \mathbf{Z}) \cong \varprojlim H^i(G_k, \mathbf{Z}) . \end{aligned}$$

Combining these two results, there exists for a representation

$$\varphi : G \to Gl_n(\mathbf{C})$$

of a locally finite group $G = \varinjlim G_k$ a unique cohomology class $c.(\varphi) \in H^{**}(G, \mathbf{Z})$ such that for all k

$$\operatorname{res}^G_{G_k}(c.(\varphi)) = c.(\varphi_k)$$

Using this uniqueness result, it is easy to see that these classes satisfy the properties CH1, CH2 and CH3 and that they are uniquely determined by these properties.

REFERENCES

[1] Atiyah, M. F. Characters and cohomology of finite groups. *Inst. Hautes Etudes Sci. Publ. Math. 9* (1961), 2-64.

[2] Evens, L. On the Chern classes of representations of finite groups. *Trans. Amer. Math. Soc. 115* (1965), 180-193.

[3] Evens, L. and D. Khan. An integral Riemann-Roch formula for induced representations of finite groups. *Trans. Amer. Math. Soc. 245* (1978), 309-330.

[4] Green, J. A. The characters of the finite general linear groups. *Trans. Amer. Math. Soc. 80* (1955), 402-447.

[5] Grothendieck, A. Classes de Chern des représentations linéaires des groupes discrets. In *Dix Exposés sur la Cohomologie étale des schémas,* North Holland, 1968.

[6] Kroll, O. The cohomology of the finite general linear group. To appear in *J. Pure and Appl. Algebra.*

[7] —— An algebraic characterisation of Chern classes of finite group representations. *Bull. London Math. Soc. 19* (1987), 245-248.

[8] QUILLEN, D. Characteristic classes of representations. In *Algebraic K-Theory*, Evanston, 1976. Springer Verlag, 1976.
[9] SERRE, J.-P. *Linear representations of finite groups*. Springer Verlag, 1977.
[10] THOMAS, C. B. Chern classes of representations. *Bull. London Math. Soc. 18* (1986), 225-240.

(Reçu le 15 février 1988)

Ove Kroll

Matematisk Institut
Aarhus Universitet
Ny Munkegade — Bygning 530
DK-8000 Aarhus (Denmark)