§5. Representation masses in Z

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For every finite set S of primes and for every integer $s \geqslant 1$ we consider the following function defined on $\mathbf{A}$:

$$r_{S,s}(n, f, \mathbf{A}) = r(n_\infty, f_\infty, \mathbf{R}) \,.\, \prod_{p \in S} r_s(n_p, f_p, \mathbf{Z}_p) \,.\, \prod_{p \notin S} r(n_p, f_p, \mathbf{Z}_p)\,.$$

As before, $r_{S,s}$ is well-defined, continuous and integrable if $k \geqslant 5$. The corresponding function $\theta_{S,s}(\ , f, \mathbf{A})$ will be well-defined and continuous for all $k \geqslant 1$, being the Fourier transform of the former.

Since $f \sim g$ over $\mathbf{A}$ is equivalent to $f_p \sim g_p$ over $\mathbf{Q}_p$ for all p including $p = \infty$, and $f_p \sim g_p$ over $\mathbf{Z}_p$ for almost all p, we get from Theorem 2.3, 3.3 and 3.4 the following:

THEOREM 4.2. *Let f, g be two non-singular integral adelic quadratic forms in $k \geqslant 5$ variables. Let $S = \{p\,; p \mid \det f_p \,.\, \det g_p\}$ and let $s \geqslant \max(s_o(f), s_o(g))$. Then the following conditions are equivalent:*

i) $f \sim g$ *over* $\mathbf{A}$,

ii) $r_{S,s}(\ , f, \mathbf{A}) = r_{S,s}(\ , g, \mathbf{A})$,

iii) $\theta_{S,s}(\ , f, \mathbf{A}) = \theta_{S,s}(\ , g, \mathbf{A})$. $\square$

Note that we could have also expressed these functions as $r_{S,s} = r_{\Phi_{S,s}}$, $\theta_{S,s} = \theta_{\Phi_{S,s}}$, where $\Phi_{S,s} \in L^1(\mathbf{A}^k)$ is defined as:

$$\Phi_{S,s} = \phi_\infty \cdot \prod_{p \in S} \phi_s \cdot \prod_{p \notin S} 1_{(\mathbf{Z}_p)^k}\,.$$

§ 5. Representation masses in $\mathbf{Z}$

Let (V, q) be a regular quadratic space over $\mathbf{Q}$ of dimension k, and let G be the proper orthogonal group of this space. The adele group $G(\mathbf{A})$ operates in the set of lattices L of V; by definition the orbit of L under this action is called the genus of L. The orbit of L under the subgroup $G(\mathbf{Q})$ of $G(\mathbf{A})$ is the class of L.

If $L = \mathbf{Z}e_1 \oplus \ldots \oplus \mathbf{Z}e_k$ is a lattice of V, the formula

$$f(x_1, \ldots, x_k) = q(x_1 e_1 + \ldots + x_k e_k)$$

stablishes a one to one correspondence between the set of classes of lattices of (V, q) and the set of classes, over $\mathbf{Z}$, of quadratic forms which are $\mathbf{Q}$-equivalent to q.

For any $n \in \mathbf{Q}^*$, a representation of n by L is a couple (x, L) such that $x \in L$ and $q(x) = n$. Since the groups $G(\mathbf{Q})$, $G(\mathbf{A})$ operate on the set of such representations, one can group them in classes and genera, respectively.

For each $x \in q^{-1}(n)$, the stabilizer of G at x can be identified with the orthogonal group of the quadratic form induced by q on $<x>^{\perp}$. By Witt's theorem, the action of $G(\mathbf{Q})$ on $q^{-1}(n)$ is transitive. Suppose that $q^{-1}(n) \cap L \neq \emptyset$ and fix $x_o \in q^{-1}(n) \cap L$. Let us choose gauge forms ω on G and ω_{x_o} on g_{x_o}. If $\sigma \in G(\mathbf{Q})$ and $x = \sigma x_o$, we consider on $g_x = \sigma g_{x_o} \sigma^{-1}$ the gauge forms ω_x obtained from ω_{x_o} by pull back. Let μ_∞, μ_p, μ; $\mu_{x,\infty}, \mu_{x,p}, \mu_x$ be the respective local measures and Tamagawa measure induced by these gauge forms on G and g_x. The homogeneous space G/g_x can be identified with $q^{-1}(n)$ and there exists a unique gauge form ω' on $q^{-1}(n)$ such that if $\mu'_\infty, \mu'_p, \mu'$ denote the local measures and Tamagawa measure induced by ω', then $\mu = \mu_x \,.\, \mu'$ (cf. [18]).

The representation mass of n by (x, L) is defined in [6] as:

$$r(n, (x, L)) = \mu_{x,\infty}\big(g_x(\mathbf{R})/g_x(\mathbf{Q}) \cap G_L\big),$$

where G_L is the stabilizer of the lattice L in $G(\mathbf{Q})$. By the above normalization of gauge forms, this definition depends only of the class of (x, L). Thus, one can define the representation mass of n by L as

$$r(n, L) = \sum_x r(n, (x, L)),$$

x running over a system of representatives of the classes (x, L) with fixed L.

Let $L_1, \ldots, L_h$ be a system of representatives of the classes in the genus of L and let $G_i = G_{L_i}$. The mass of the genus of L is defined as

$$m(\text{gen}\, L) = \sum_{i=1}^{h} \mu_\infty\big(G(\mathbf{R})/G_i\big),$$

and the representation mass of n by the genus of L as

$$r(n, \text{gen}\, L) = m(\text{gen}\, L)^{-1} \sum_{i=1}^{h} r(n, L_i).$$

If q is definite and $k \geqslant 3$ or if q is indefinite and $k \geqslant 4$, the Tamagawa number of G and g_x is 2. From this fact it can be deduced (cf. [17]) that

$$r(n, \text{gen}\, L) = \prod_p \mu'_p\big(q^{-1}(n) \cap L_p\big),$$

where L_p is the localization of L at p.

From now on we shall assume that f is a $\mathbf{Z}$-integral quadratic form *positive definite in* $k \geqslant 3$ *variables or indefinite in* $k \geqslant 4$ *variables.*

Let $L = \mathbf{Z}^k$. By normalizing ω or ω_{x_o}, we can assume that

$$\mu'_p(f^{-1}(n) \cap \mathbf{Z}_p^k) = r(n, f, \mathbf{Z}_p).$$

Therefore, we obtain Siegel's formula:

$$r(n, \operatorname{gen} \mathbf{Z}^k) = \prod_p r(n, f, \mathbf{Z}_p). \tag{4.1}$$

The number on the left of (4.1) admits a quite natural interpretation in the definite case, due to the fact that the set $f^{-1}(n) \cap \mathbf{Z}^k$, the group $G_{\mathbf{Z}^k}$, and the three volumes which appear in the formula

$$\mu_\infty(G(\mathbf{R})) = \mu_{x,\infty}(g_x(\mathbf{R})) \,.\, \mu'_\infty(f^{-1}(n))$$

are all finite. In fact, defining $r(n, f) = \#(f^{-1}(n) \cap \mathbf{Z}^k)$, $o(f) = \# G_{\mathbf{Z}^k}$, and denoting by $f_1, \ldots, f_h$ a complete system of representatives of the classes of forms in the genus of f, from the above considerations it is not hard to deduce the following set of formulas in the definite case:

$$r(n, \mathbf{Z}^k) = \frac{\mu_\infty(G(\mathbf{R}))}{\mu'_\infty(f^{-1}(n))} \cdot \frac{r(n, f)}{o(f)},$$

$$m(\operatorname{gen} \mathbf{Z}^k) = \mu_\infty(G(\mathbf{R})) \sum_{i=1}^{h} o(f_i)^{-1},$$

$$r(n, \operatorname{gen} \mathbf{Z}^k) = \mu'_\infty(f^{-1}(n))^{-1} \Big(\sum_{i=1}^{h} r(n, f_i) o(f_i)^{-1}\Big) / \Big(\sum_{i=1}^{h} o(f_i)^{-1}\Big).$$

Moreover, the factor $\mu'_\infty(f^{-1}(n))$ is, by definition, equal to the function $F_\Phi(n, f, \mathbf{R})$ for $\Phi = 1$ (see the end of Section 2). Hence, we have

$$\mu'_\infty(f^{-1}(n)) = \lim_{U \to \{n\}} \big(\operatorname{vol}(f^{-1}(U))/\operatorname{vol} U\big).$$

We recover in this way Siegel's real density of representations [13], which has the well-known value:

$$\mu'_\infty(f^{-1}(n)) = \pi^{k/2}\Gamma(k/2)^{-1} (\det f)^{-1/2} n^{k/2-1}$$

if $n > 0$ (if $n < 0$ is $\mu'_\infty(f^{-1}(n)) = 0$).

In order to be coherent with the classical notation, we define the integral representation masses $r(n, f)$, $r(n, \operatorname{gen} f)$ in a different way, according f to be definite or indefinite.

$$r(n, f) := \begin{cases} \#(f^{-1}(n) \cap \mathbf{Z}^k) & \text{if} \quad f \text{ definite} \\ r(n, \mathbf{Z}^k) & \text{if} \quad f \text{ indefinite} \end{cases}$$

$$r(n, \text{gen } f) := \begin{cases} r(n, \text{gen } \mathbf{Z}^k) \,.\, \mu'_\infty(f^{-1}(n)) & \text{if} \quad f \text{ definite} \\ r(n, \text{gen } \mathbf{Z}^k) & \text{if} \quad f \text{ indefinite}\,. \end{cases}$$

Let us denote, moreover, $\mu(f) = \mu_\infty(G(\mathbf{R})/G_{\mathbf{Z}^k})$.

It is a well-known fact that, in the indefinite case, and for all $n \in \mathbf{Z}\backslash\{0\}$

$$r(n, \text{gen } f) = \mu(f)^{-1} r(n, f)\,,$$

since the average representation mass in a spinor genus coincides with $r(n, \text{gen } f)$, but for $k \geqslant 4$ there is only one class in each spinor genus.

Summing up all this considerations we can rewrite Siegel's formula in the form:

$$r(n, \text{gen } f) = \mu'_\infty(f^{-1}(n)) \,.\, \prod_p r(n, f, \mathbf{Z}_p)$$

if f is definite,

$$r(n, f) = \mu(f) \,.\, \prod_p r(n, f, \mathbf{Z}_p)\,,$$

otherwise.

We can now reproduce partially the outline of the preceding sections. Considering $r(\ , f)$, $r(\ , \text{gen } f)$ as functions defined on $\mathbf{Z}$, we can define theta series by taking the formal Fourier transform:

$$\theta(z, f) = \sum_{n \geqslant 0} r(n, f) \exp(\pi i n z)\,,$$

$$\theta(z, \text{gen } f) = \sum_{n \geqslant 0} r(n, \text{gen } f) \exp(\pi i n z)\,;$$

and zeta functions by taking formal Mellin transforms:

$$\zeta(s, f) = \sum_{n > 0} r(n, f) n^{-s}\,,$$

$$\zeta(s, \text{gen } f) = \sum_{n > 0} r(n, \text{gen } f) n^{-s}\,.$$

Both functions have been largely investigated. We recall next their more relevant properties for our purposes (cf. [13], [14], [15], [12], [10]). If f is definite, $\theta(z, f)$ is a modular form of weight $k/2$, with character, with respect to a congruence group $\Gamma_o(N)$. It satisfies the functional equation

$$(5.1) \qquad \theta(z, f) = (\det f)^{-1/2}(-iz)^{-k/2}\theta(-1/z, f^\sharp)\,,$$

where $f^\sharp$ denotes the quadratic form associated to the dual lattice of $\mathbf{Z}^k$

in $\mathbf{R}^k$ with respect to f. And $\theta(z, \operatorname{gen} f)$ is an Eisenstein series for the same group.

The Dirichlet series defining $\zeta(s, f)$ converges, both in the definite and in the indefinite case, for $\operatorname{Re} s > k/2$. It has a meromorphic continuation to the whole plane with a simple pole at $s = k/2$ (and possibly at $s = 1$, if f is indefinite) and it satisfies a functional equation involving $\zeta(s, f)$ and $\zeta(k/2 - s, f^{-1})$. Clearly the zeta function $\zeta(\ , \operatorname{gen} f)$ has the same properties.

In the indefinite case, the residue at $s = k/2$ of these zeta functions is given by:

$$[\zeta(s, f)]_{k/2} = 2\rho_k \,|\det f\,|^{k/2} \mu(f)\,, \tag{5.2}$$

$$[\zeta(s, \operatorname{gen} f)]_{k/2} = 2\rho_k \,|\det f\,|^{k/2}\,, \tag{5.3}$$

where

$$\rho_k := \sum_{j=1}^{k-1} \Gamma(j/2)\pi^{-j/2}\,.$$

THEOREM 5.1. *Let f, g be two non singular $\mathbf{Z}$-integral quadratic forms. Suppose that $f \sim g$ over $\mathbf{R}$ and that they are of the same 2-type if $k \geqslant 5$. Then the following conditions are equivalent:*

i) $\operatorname{gen} f = \operatorname{gen} g$,

ii) $r(\ , \operatorname{gen} f) = r(\ , \operatorname{gen} g)$,

iii) $\zeta(\ , \operatorname{gen} f) = \zeta(\ , \operatorname{gen} g)$.

Proof. It is clear from the definitions that i) $\Rightarrow$ ii) $\Rightarrow$ iii). Assume that iii) is satisfied and let us show first that it must be $\det f = \det g$. In the indefinite case, this is a direct consequence of (5.3) and the fact that $f \sim g$ over $\mathbf{R}$. In the definite case, and since iii) is equivalent to the equality $\theta(\ , \operatorname{gen} f) = \theta(\ , \operatorname{gen} g)$, by (5.1) we have

$$(\det f)^{-1/2}\, \theta(\ , \operatorname{gen} f^{\#}) = (\det g)^{-1/2}\, \theta(\ , \operatorname{gen} g^{\#})\,.$$

Since $f^{\#}, g^{\#}$ are two definite quadratic forms we have

$$\lim_{t \to \infty} \theta(it, \operatorname{gen} f^{\#}) = \lim_{t \to \infty} \theta(it, \operatorname{gen} g^{\#}) = 1\,,$$

hence $\det f = \det g$ and, moreover, $\mu'_\infty(f^{-1}(n)) = \mu'_\infty(g^{-1}(n))$.

By Siegel's formula, we see that condition iii) implies, in both cases, that

$$\prod_p r(n, f, \mathbf{Z}_p) = \prod_p r(n, g, \mathbf{Z}_p)$$

for all $n \neq 0$. Let now S be any finite set of primes including those p dividing $2 \det f$. Assume $n \in \mathbf{Z}\backslash\{0\}$. If $p \notin S$ we have by [13, Hilfsatz 16] that

$$r(n, f, \mathbf{Z}_p) = r(n, g, \mathbf{Z}_p) \neq 0 .$$

Therefore, by [13, Hilfsatz 25] we have:

$$\prod_{p \in S} r(n, f, \mathbf{Z}_p) = \prod_{p \in S} r(n, g, \mathbf{Z}_p) .$$

Let $\mathbf{Z}_S = \prod_{p \in S} \mathbf{Z}_p$. By the chinese remainder theorem we get the equality of functions over $\mathbf{Z}_S$:

$$\prod_{p \in S} r(\ , f, \mathbf{Z}_p) = \prod_{p \in S} r(\ , g, \mathbf{Z}_p) .$$

Since

$$\begin{aligned}\prod_{p \in S} \theta(m_p, f, \mathbf{Z}_p) &= \prod_{p \in S} \int_{\mathbf{Z}_p} r(n_p, f, \mathbf{Z}_p) <n_p, m_p> dn_p \\ &= \int_{\mathbf{Z}_S} (\prod_{p \in S} r(n_p, f, \mathbf{Z}_p) <n_p, m_p>) (\bigotimes_{p \in S} dn_p) \\ &= \int_{\mathbf{Z}_S} (\prod_{p \in S} r(n_p, f, \mathbf{Z}_p)) <n_S, m_S> dn_S ,\end{aligned}$$

where n_S, m_S, dn_S have their natural meanings, we see that condition iii) implies

$$\prod_{p \in S} \theta(\ , f, \mathbf{Q}_p) = \prod_{p \in S} \theta(\ , g, \mathbf{Q}_p) .$$

Taking into account that $\theta(\mathbf{Z}_p, f, \mathbf{Q}_p) = 1$, we get that $\theta(\ , f, \mathbf{Q}_p) = \theta(\ , g, \mathbf{Q}_p)$, for all $p \in S$. Thus, by applying Theorem 2.3 we get that $f \sim g$ over $\mathbf{Z}_p$, for all p. $\square$

We have proved that the representation mass function $r(\ , \mathrm{gen}\, f)$ determines the genus of f under certain conditions on the ∞-type and the 2-type of f. The following examples of forms f, g such that $r(\ , \mathrm{gen}\, f) = r(\ , \mathrm{gen}\, g)$ but not belonging to the same genus, show that none of these conditions can be dropped (cf. also [5]).

Examples. We consider $I = \perp_{1 \leqslant i \leqslant 4} <1>$, $J = \perp_{1 \leqslant i \leqslant 4} <-1>$. Let $f = I \perp I \perp J$ and $g = I \perp J \perp J$. These two forms satisfy $f \sim g$ over $\mathbf{Z}_p$ for all p, but they are not $\mathbf{R}$-equivalent. Let $f = <1, 1, 2, 4, 4>$,

$g = \langle 1, 2, 2, 2, 4 \rangle$ or $f = \langle -1, 1, 2, 4, 4 \rangle$ and $g = \langle -1, 2, 2, 2, 4 \rangle$. In both cases f and g are $\mathbf{R}$-equivalent and satisfy $r(\ , \operatorname{gen} f) = r(\ , \operatorname{gen} g)$, but they are not $\mathbf{Z}_2$-equivalent.

In the following theorem we show that, both in the definite and in the indefinite case, two quadratic forms with the same representation numbers must belong to the same genus. In the low dimensional cases ($k=2$ or $k=3$, indefinite) an analogous result can also be stated. If $k = 3$ and the forms are indefinite, the proof requires a finer study of their representation masses (cf. [11]). If $k = 2$ much more is true, since two $\mathbf{Z}$-integral quadratic forms with the same 2-type which represent the same set of integers belong already to the same genus.

THEOREM 5.2. *Let f, g be two non-singular $\mathbf{Z}$-integral quadratic forms in k variables. Suppose that $f \sim g$ over $\mathbf{R}$ and that f and g are of the same 2-type if $k \geqslant 5$. Then $r(\ , f) = r(\ , g)$ implies that f and g belong to the same genus.*

Proof. $k \geqslant 3$, *f definite.* By hypothesis we have $\theta(\ , f) = \theta(\ , g)$ as functions on the upper half-plane. As is well-known, $\theta(\ , f) - \theta(\ , \operatorname{gen} f)$ is a cusp form. Thus $\theta(\ , \operatorname{gen} f) - \theta(\ , \operatorname{gen} g)$, being both a cusp form and an Eisenstein series, must be zero. Applying Theorem 5.1 we have that $\operatorname{gen} f = \operatorname{gen} g$.

$k \geqslant 4$, *f indefinite.* Since $r(\ , \operatorname{gen} f) = \mu(f)^{-1} r(\ , f)$, we need only to show that $\mu(f) = \mu(g)$ and apply Theorem 5.1. By hypothesis $\zeta(\ , f) = \zeta(\ , g)$; from the residue formula (5.2) we get

$$| \det f |^{-1/2} \mu(f) = | \det g |^{-1/2} \mu(g) . \tag{5.4}$$

There is an explicit relation between $\mu(f)$ and the volume $V(f)$ of the majorante space [16, p. 110] which, together with the fact $V(f) = V(f^{-1})$ furnishes the relation

$$\mu(f^{-1}) = | \det f |^{k+1} \mu(f) .$$

Now, from the functional equation of the zeta function [14], it is easily deduced that $\mu(f^{-1}) = \mu(g^{-1})$, hence

$$| \det f |^{k+1} \mu(f) = | \det g |^{k+1} \mu(g) .$$

This together with (5.4) implies $| \det f | = | \det g |$ and $\mu(f) = \mu(g)$. □