E) Units

Indeed,

$$(2, \omega)(2, \omega') = \left(4, 2\omega, 2\omega', \frac{1-d}{4}\right) = A2\left(2, \omega, \omega', \frac{1-d}{8}\right) = A2,$$

because $\omega + \omega' = 1$.

Also $(2, \omega) \neq (2, \omega')$, otherwise these ideals are equal to their sum $(2, \omega, \omega') = A$, because $\omega + \omega' = 1$.

c) If $d \equiv 2$ or $3 \pmod 4$ then $A2 = (2, \sqrt{d})^2$, respectively $(2, 1+\sqrt{d})^2$. First let $d = 4e + 2$ then

$$(2, \sqrt{d})^2 = (4, 2\sqrt{d}, d) = A2(2, \sqrt{d}, 2e+1) = A2,$$

so $(2, \sqrt{d})$ is a prime ideal.

Now, let $d = 4e + 3$, then

$$\begin{aligned}(2, 1+\sqrt{d})^2 &= (4, 2+2\sqrt{d}, 1+d+2\sqrt{d}) = (4, 2+2\sqrt{d}, 4(e+1)+2\sqrt{d}) \\ &= A2(2, 1+\sqrt{d}, 2(e+1)+\sqrt{d}) = A2(2, 2e+1, 1+\sqrt{d}, 2(e+1)+\sqrt{d}) = A2\end{aligned}$$

and so $(2, 1+\sqrt{d})$ is a prime ideal.

Finally, these three cases are exclusive and exhaustive, so the converse assertions also hold. □

E) Units

The element $\alpha \in A$ is a unit if there exists $\beta \in A$ such that $\alpha\beta = 1$. The set U of units is a group under multiplication. Here is a description of the group of units in the various cases. First let $d < 0$.

Let $d \neq -1, -3$. Then $U = \{\pm 1\}$.

Let $d = -1$. Then $U = \{\pm 1, \pm i\}$, with $i = \sqrt{-1}$.

Let $d = -3$. Then $U = \{\pm 1, \pm \rho, \pm \rho^2\}$, with $\rho^3 = 1$, $\rho \neq 1$, i.e. $\rho = \dfrac{-1 + \sqrt{-3}}{2}$.

Let $d > 0$. Then the group of units is the product $U = \{\pm 1\} \times C$, where C is a multiplicative cyclic group. Thus $C = \{\varepsilon^n \mid n \in \mathbf{Z}\}$, where ε is the smallest unit such that $\varepsilon > 1$. ε is called the fundamental unit.