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THE THEOREM OF BROWN AND SARD

by Per HOLM

There exists several proofs of the Brown-Sard theorem on regular values of differentiable mappings, [1], [3]. If one keeps to the infinitely differentiable case, perhaps the most elegant is the one given in Milnor's book "Topology from the differentiable point of view", [2]; it has become prototype for many expositions. In this paper we give a variant of the proof which is simpler and more transparent. Briefly, it consists of a proof for the case of functions $X \to \mathbf{R}$ and a splitting argument which reduces the case of mappings $X \to \mathbf{R}^p$ to the case $X \to \mathbf{R}^{p-1}$ for arbitrary manifolds X. This makes a point in favour of first introducing the theorem for functions, and then later, by the splitting argument, extend it to the general case. In introductory courses on singularities of mappings (for instance) this is a very natural procedure.

1. All manifolds are assumed to be smooth (C^∞) and with countably based topology. If $f : X \to Y$ is a smooth mapping, we denote by C_f the set of critical points of f, by D_f the set of critical values, and by R_f the set of regular values (so $R_f = Y - D_f$). We note that C_f is a closed subset of X.

Our aim is to prove the following version of the theorem of Brown and Sard:

THEOREM. *Let X be an n-dimensional manifold. The set of regular values of any smooth mapping $f : X \to \mathbf{R}^p$ is a countable intersection of open dense subsets of $\mathbf{R}^p$. Equivalently, the set of critical values of f is a countable union of closed subsets with empty interior.*

The theorem has an easy extension to the case where $\mathbf{R}^p$ is replaced by an arbitrary p-manifold Y. On the other hand it can be deduced from a more special case:

Corollary. *The regular values of a smooth mapping* $f: \mathbf{R}^n \to \mathbf{R}^p$ *form a dense subset of* $\mathbf{R}^p$. *Equivalently, the critical values of* f *form a set with empty interior.*

To see that the theorem can be deduced from its corollary, consider the composite mapping $f \circ \phi: \mathbf{R}^n \to \mathbf{R}^p$, where $\phi: \mathbf{R}^n \to X$ is a local parametrization of X. Set $W = \phi(\mathbf{R}^n)$ and $K = \phi(D^n)$, where D^n is the closed unit ball in $\mathbf{R}^n$. By the corollary the set $D_{f\circ\phi} = f(C_f \cap W)$ has empty interior. Thus $f(C_f \cap K)$ is a compact subset of $\mathbf{R}^p$ with empty interior. Since X can be covered by countably many ball neighbourhoods K, $f(C_f)$ is a countable union of compact subsets with empty interior. This yields the theorem. Moreover the same type of argument extends it to the case where $\mathbf{R}^p$ is replaced by an arbitrary p-manifold.

2. The case of functions ($p=1$). We just observed it suffices to consider functions on the coordinate space $\mathbf{R}^n$. If $n = 0$, there is nothing to prove. Assume the case $n-1$ settled and consider a smooth function f on $\mathbf{R}^n$. Let $C_i \subset \mathbf{R}^n$ be the closed set of points where all partial derivatives of f of order $\leqslant i$ vanish. Then $C_f = C_1$ and $C_1 \supseteq C_2 \supseteq \cdots$. Hence

$$C_f = (C_1 - C_2) \cup \cdots \cup (C_{n-1} - C_n) \cup C_n,$$

and so

$$D_f = f(C_1 - C_2) \cup \cdots \cup f(C_{n-1} - C_n) \cup f(C_n).$$

Therefore the theorem follows from the sublemmas 1 and 2 below.

Sublemma 1. $f(C_i - C_{i+1})$ *is a countable union of closed sets without interior points,* $i \geqslant 1$.

Proof. We claim the following: For each $u \in C_i - C_{i+1}$ there is a compact neighbourhood K disjoint with C_{i+1} and an $(n-1)$-dimensional submanifold Z such that $C_i \cap K \subseteq Z$.

Then every point of $C_i \cap K$ is critical for $f \mid Z$, since it is critical for f. By our induction assumption $f(C_i \cap K)$ is then a closed set without interior points in $\mathbf{R}$. Since $C_i - C_{i+1}$ can be covered by countably many arbitrarily small compact neighbourhoods K, we obtain sublemma 1.

As for the claim, since $u \in C_i - C_{i+1}$ there is some i-th order partial derivative g of f whose first order partial derivatives do not all vanish at u. Then u is a regular point for g and $g(u) = 0$. Hence $g^{-1}\{0\}$ is an $(n-1)$-dimensional manifold in some open neighbourhood U of u. Moreover

$C_i \subseteq g^{-1}\{0\}$. Set $Z = g^{-1}\{0\} \cap U$ and let K be any sufficiently small neighbourhood of u in U.

SUBLEMMA 2. *$f(C_i)$ is a countable union of closed sets without interior points, $i \geqslant n$.*

Proof. By Taylor's theorem we have

$$f(x+u) = f(x) + D_u f(x) + \cdots + \frac{1}{i!} D_u^i f(x) + \frac{1}{(i+1)!} D_u^{i+1} f(x+\lambda u)$$

for any two points x and u in $\mathbf{R}^n$, where D_u is the differential operator $D_u = u_1\partial/\partial x_1 + \cdots + u_n \partial/\partial x_n$ and λ is some real number between 0 and 1. Thus if $x \in C_i$, then $f(x+u) - f(x) = 1/(i+1)! D_u^{i+1} f(x+\lambda u)$. If in addition x and $y = x + u$ is confined to a convex open set K, then $x + \lambda u$ is also in K, and we get the inequality

$$| f(y) - f(x) | \leqslant c \, | y - x |^{i+1}$$

where $| u |$ means $\max \{| u_1 |, \cdots, | u_n |\}$ and c is a constant depending on K and f only.

Now take K to be a unit cube in $\mathbf{R}^n$ and consider the subdivision of K into k^n subcubes of sidelength $1/k$. Let K' be one of these and suppose $x \in C_i \cap K'$, $y \in K'$. Then $| y - x | \leqslant 1/k$, showing that $f(C_i \cap K')$ is contained in an interval of length c/k^{i+1}. Consequently $f(C_i \cap K)$ is contained in a union of k^n intervals of joint length $k^n c/k^{i+1} \leqslant c/k$. Since here k is any positive integer, this length can be arbitrarily small, and so the set $f(C_i \cap K)$ must have empty interior. Finally $\mathbf{R}^n$ and therefore C_i is contained in a countable union of unit cubes K.

3. THE GENERAL CASE. Assume the theorem holds for mappings into $\mathbf{R}^{p-1}$, and consider a smooth mapping $f: X \to \mathbf{R}^p (p \geqslant 2)$. We shall prove that the regular values of f are dense in $\mathbf{R}^p$. (The formulation given in the corollary.)

So let $O \subseteq \mathbf{R}^p$ be any nonempty open subset of $\mathbf{R}^p$. We will show that f has regular values in O. Since there is nothing to prove if O sticks outside $f(\mathbf{R}^n)$, assume $O \subseteq f(\mathbf{R}^n)$.

Let $\pi: \mathbf{R}^p \to \mathbf{R}^{p-1}$ be the projection; then $\pi(O)$ is open in $\mathbf{R}^{p-1}$. By our induction hypothesis the mapping $\pi \circ f$ has a regular value $y' \in \pi(O)$; equivalently f is transverse to the line $Y' = \pi^{-1}\{y'\}$ in $\mathbf{R}^p$.

Now set $X' = f^{-1}(Y')$ and let $f': X' \to Y'$ be the induced mapping. The open set O meets $Y' \cong \mathbf{R}$, and so f' has a regular value $y'' \in Y' \cap O$

by the Brown-Sard theorem for functions (case 1). Altogether, f is transverse to Y' in $\mathbf{R}^p$ and f' has y'' as regular value in Y'. But then f has y'' as regular value in $\mathbf{R}^p$.

Remark. At the end we used the rather trivial fact that if $f: X \to \mathbf{R}^p$ is transverse to a submanifold $Y' \subseteq \mathbf{R}^p$ and the induced mapping $f': X' \to Y'$ is transverse to another $Y'' \subseteq Y'$, then f is transverse to Y'' (the splitting argument). Apart from that we used the fact that for a smooth function any solution set $f^{-1}\{y\}$ is a codimension 1 submanifold near any regular point (the regular point property). And of course we repeatedly used the Baire category theorem.

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