I. Review of the Newton Polygon

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.09.2024**

L'Enseignement Mathématique, t. 33 (1987), p. 183-189

ON THE GALOIS GROUPS OF THE EXPONENTIAL TAYLOR POLYNOMIALS

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Let

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

denote the n^{th} Taylor polynomial of the Exponential Function.

In 1930, [S-2], Schur proved the following theorem about $f_n(x)$:

THEOREM (Schur). *The Galois group, G_n, of f_n is A_n, the alternating group on n letters, if 4 divides n and is S_n, the symmetric group on n letters, otherwise.*

In this note we shall give a proof different from Schur's. Our ulterior motive is to demonstrate the utility of Newton polygons.

We must:

A. Show that f_n is irreducible.

B. Show that G_n contains a p-cycle for any prime number p between $n/2$ and $n-2$.

C. Calculate the discriminant, D_n, of f_n and determine when it is a square.

We need the following:

1. The main theorem about p-adic Newton polygons (see below).
2. Bertrand's Postulate [B], proven by Tschebyshev [T], which asserts that for each integer n, at least 8, there exists a prime number strictly between $n/2$ and $n-2$. (See also [H-W] Chapter 22.)
3. The theorem of Jordan which asserts that if G is a transitive subgroup of S_n which contains a p-cycle for some prime p strictly between $n/2$ and $n-2$ then G contains A_n. (See [J-1], Note C and [J-2], Theorem 1 or [Ha], Theorems 5.6.2 and 5.7.2.)
4. The fact that the Galois group of a polynomial of degree n is contained in A_n iff its discriminant is a square.

We shall use 1 for A and B. This will imply G_n is transitive and together with 2 and 3 will imply G_n contains A_n for $n \geqslant 8$. We shall use the differential equation satisfied by the exponential function and 2 again to perform C. Finally, we shall use 4 to complete the proof. (We shall also require liberal doses of Galois theory.)

I. Review of the Newton Polygon

Let

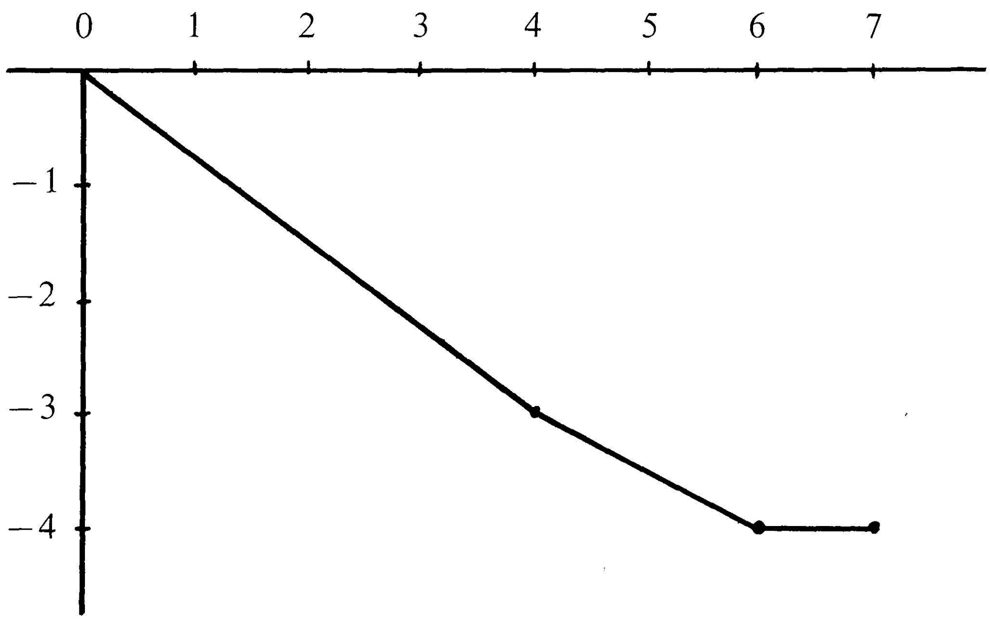
$$g(x) = a_0 + a_1 x + \dots + a_k x^k$$

be a polynomial over $\mathbf{Q}_p$. Consider the points:

$$(i, \operatorname{ord}(a_i)), \quad 0 \leqslant i \leqslant k,$$

in the Cartesian plane. The Newton polygon of g is defined to be the lower convex hull of these points.

Example. The Newton Polygon of $f_7(x)$ considered over $\mathbf{Q}_2$, is



The main theorem about these polygons is:

THEOREM NP. *Let* $(x_0, y_0), (x_1, y_1), \dots, (x_p, y_p)$ *denote the successive vertices of this polygon. Then over* $\mathbf{Q}_p$, g *factors as follows:*

$$g(x) = g_1(x) g_2(x) \dots g_r(x)$$

where the degree of g_i *is* $x_i - x_{i-1}$ *and all the roots of* $g_i(x)$ *in* $\bar{\mathbf{Q}}_p$ *have valuation* $-\left(\dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}\right)$.

We call the rational numbers, $\dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$, the slopes of g.

Example. The polynomial f_7 has three factors over $\mathbf{Q}_2$, of degrees 4, 2 and 1, respectively, which have slopes $-3/4$, $-1/2$ and 0.

COROLLARY. *Let* d *be a positive integer. Suppose that* d *divides the denominator of each slope (in lowest terms) of* g. *Then* d *divides the degree of each factor of* g *over* $\mathbf{Q}_p$.

Proof. It suffices to show that d divides the degree of each irreducible factor of g. Let h be such a factor. Let $\alpha \in \bar{\mathbf{Q}}_p$ be a root of h. Since d divides the denominator of the valuation of α (by Theorem NP), it follows that d divides the index of ramification of the extension $\mathbf{Q}_p(\alpha)/\mathbf{Q}_p$ which divides the degree of the extension which equals the degree of h.

II. APPLICATION TO THE EXPONENTIAL TAYLOR POLYNOMIALS

Fix a prime number p.

LEMMA. *Suppose* k *is a positive integer and*

$$k = a_0 + a_1 p + \dots a_s p^s$$

where $0 \leqslant a_i < p$. *Then*

$$\operatorname{ord}(k!) = \frac{k - (a_0 + a_1 + \dots + a_s)}{p - 1}.$$

This is easy and well known.

Now write

$$n = b_1 p^{n_1} + b_2 p^{n_2} + \dots + b_s p^{n_s}$$

where $n_1 > n_2 > \dots > n_s$ and $0 < b_i < p$. Let

$$x_i = b_1 p^{n_1} + \dots + b_i p^{n_i}.$$