2. Kähler-Einstein Metrics on Compact Kähler Manifolds

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In general, one would like to compare the Bergman, Kobayashi-Royden, Caratheodory metrics and the Kähler-Einstein metric discussed in the next section. We know that the Caratheodory metric is the smallest of the three. This can be seen by using the generalized Schwarz lemma for Kähler manifolds [Y4]. Yau (see the later improvement by Chan-Cheng-Lu) proved that if $f: M \to N$ is a holomorphic map where M is a complete Kähler manifold with Ricci curvature bounded from below by a constant and N is a Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant, then f decreases distances up to a constant depending on the curvatures of M and N. Is this true if N is only a Finsler space? If it were true, then one expects that Teichmüller metric is uniformly equivalent to the Kähler-Einstein metric.

2. Kähler-Einstein Metrics on Compact Kähler Manifolds

Let M be a compact Kähler manifold. A necessary condition for the existence of a Kähler-Einstein metric on M is as follows.

(*) There exists a Kähler class Ω such that the first Chern class $c_1(M)$ is cohomologous to some real constant multiple of Ω.

This condition is equivalent to the following:

(*)′ The first Chern class satisfies $c_1(M) > 0$, $c_1(M) = 0$ or $c_1(M) < 0$.

It was proved by the author [Y1], [Y2] that when $c_1(M) = 0$ or $c_1(M) < 0$, (for the latter case see also Aubin [Au3]) there exists in every Kähler class a unique Kähler-Einstein metric. When $c_1(M) > 0$, the space Kähler-Einstein metrics are invariant under automorphism group. However, existence does not hold in general and one would like to impose conditions on M to ensure existence.

We now consider the obstruction, due to Futaki [Fu1], to the existence of Kähler-Einstein metrics when $c_1(M) > 0$; we also consider the notion of "extremal metrics" due to Calabi [Ca2]. Fix a Kähler class $\Omega = [\omega] \in H^{1,1}(M)$ on a compact Kähler manifold M and denote by H_Ω the space of all Kähler metrics with Kähler class Ω. Define the functional

$$F: H_\Omega \to \mathbf{R} \quad \text{by} \quad F:(g) \to \int_M R^2,$$

where R denotes the scalar curvature of the metric g. Calabi called a critical point of this functional an extremal metric. Any Kähler-Einstein metric minimizes $\int_M R^2$ in its Kähler class and hence is an extremal metric.

This follows from the Schwarz inequality and the fact that $\int_M R$ is equal to $c_1(M) \cup \omega^{n-1}$ evaluated on the fundamental class of M, where ω is the Kähler form of g.

Calabi proved that for an extremal metric g, the gradient vector field $X = \sum g^{i\bar{j}} \frac{\partial R}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ is holomorphic. He also proved that a decomposition theorem holds, analogous to that of Matsushima and Lichnerowicz for constant scalar curvature, for the automorphism group of M. In particular, he proved that X gives rise to a compact subgroup of Aut (M). Levine [Lv] gave an example of a compact surface M^2 with no compact connected subgroup in Aut (M); hence M^2 does not admit any Kähler-Einstein metrics.

For other examples of when Aut (M) is not reductive, see Sakane [Sk1], [Sk2], Ishikawa-Sukane [I-S] and Yau [Y3]. By the theorems of Calabi or Matsushima-Lichnerowicz, these examples do not admit any Kähler-Einstein metrics. Futaki [Fu1] also has constructed examples where Aut (M) is reductive and we will consider them later. So far, however, all examples of a Kähler manifold with positive first Chern class which does not admit a Kähler-Einstein metric admit nontrivial holomorphic vector field, it is natural to ask the following question: If there exists no nonzero holomorphic vector field on M, and if the tangent bundle of M is stable, can we always minimize the functional F? The motivation for the assumption on the stability will be discussed later. Of course, if the answer to the above question is yes, then (*) would also be a sufficient condition for the existence of Kähler-Einstein metrics.

In fact, suppose $c_1(M) = C[\omega]$ and g is an extremal metric. Since $X = \sum g^{i\bar{j}} \frac{\partial R}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ is holomorphic, it follows that $X = 0$, R is constant and the Ricci form of g is a harmonic form representing $c_1(M)$. One concludes that $R_{i\bar{j}} = Cg_{i\bar{j}}$ from the uniqueness of harmonic forms in a cohomology class; hence g is a Kähler-Einstein metric. Calabi [Ca2] proved that, each extremal metric g is a local, nondegenerate point of the functional F. The metric g also exhibits the greatest possible degree of symmetry compatible with the complex structure of M. Let C_Ω denotes the set of extremal metrics in H_Ω, which is diffeomorphic to a finite dimensional Euclidean space.

Moreover, if one metric in C_Ω has constant scalar curvature, then every metric in C_Ω has constant scalar curvature. One expects that the only critical points of F are global minimums of F, form a connected set, and that the group of automorphisms of M which preserve the class Ω acts transitively on C_Ω.

We now consider Futaki's obstruction to the existence of a Kähler-Einstein metric on compact Kähler manifold M with $c_1(M) > 0$. Let $\eta(M)$ denote the Lie algebra of holomorphic vector fields of M, ω a Kähler form representing $c_1(M)$, and γ_ω its Ricci form which also represents $c_1(M)$. Then $\gamma_\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(g_{ij})$ and hence $\gamma_\omega - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}G$ for some smooth function G. Define the character $f:\eta(M)\to\mathbf{C}$ by $f: X \to \int_M (XG)\cdot\omega^n$. Futaki proved that f is independent of the choice of representative ω of $c_1(M)$. Hence the integer $\delta_M = \dim\big(\eta(M)/\ker(f)\big)$ depends only on the complex structure of M.

If M has a Kähler-Einstein metric then $\delta_M = 0$; Futaki conjectures that the converse is also true. This would be the case if Calabi's functional F attains a minimum. Since $\gamma_\omega - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}G$, one has that $R = n + \Delta G$. Then $f(X) = \int (XG)\omega^n = \int (R^\alpha G_\alpha)\omega^n = \int |\Delta G|^2\omega^n$; hence $\delta_M = 0$ implies that $G = $ constant, i.e., g is a Kähler-Einstein metric.

Using the obstruction δ_M, Futaki gave examples of compact Kähler manifolds with $c_1(M) > 0$, Aut (M) reductive, and $\delta_M = 1$. Hence, there does not exist Kähler-Einstein metrics on these examples. Let H_n denote the hyperplane bundle of $\mathbf{CP}^n$ and $\pi_n: H_n \to \mathbf{CP}^n$ the projection map ($n=1, 2$). If we let $M^5 = \mathbf{P}(E)$ where $E = \pi_1^*(H_1) + \pi_2^*(H_2)$ is considered as a bundle over $\mathbf{CP}^2$, then M is such an example. The following is the lowest dimensional example. If $H \subseteq \mathbf{CP}^3$ is a hyperplane and $C \subseteq H$ a quadratic curve, then let M be $\mathbf{CP}^3$ blown up along C and at a point outside of H.

Futaki's idea is to construct an obstruction for the Ricci form to be harmonic. For the curvature forms representing the higher Chern classes, see Bando [B2]. For questions related to the character f, see Futaki [Fu2] and Futaki-Morita [F-M]. Bando also proved the uniqueness of Kähler-Einstein metric on M with $c_1(M) > 0$, up to holomorphic automorphisms of M.