1. Existence, uniqueness and regularity

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3. If M is a compact Riemannian manifold covered conformally by $\Omega \subset S^n$ with $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$, then M admits a metric with scalar curvature $R \geqslant 0$ in the same conformal class. It is conjectured that if $R \geqslant 0$, then $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$.

The basic idea is that by using the developing map, we can reduce the problems to the study of a scalar equation, namely the Yamabe equation on an open subset of S^n. The remaining parts of the proofs are relatively easy. By using the same technique, Schoen and Yau proved that for a compact conformally flat manifold with positive scalar curvature, $\pi_i(M) = 0$ for $2 \leqslant i \leqslant n/2$. Some of their results are also valid for complete manifolds.

§ 3. Harmonic Maps

Harmonic maps are important objects in geometry and analysis. They appear naturally as critical points of an energy functional of the appropriate function space. Harmonic maps reflect a lot about the geometric properties of manifolds.

Given Riemannian manifolds M and N, consider the mapping space $C^r(M, N)$. One problem is to find nice (i.e., canonical) representatives in this space. For a map $f : M \to N$ we define its energy by $E(f) = \int_M |df|^2 dV_M$. A harmonic map is a critical point of this energy. The first question is that of existence, uniqueness and regularity.

1. Existence, uniqueness and regularity

The first major work was done by J. Eells and L. Sampson [ES]. They proved the existence of a harmonic map in any homotopy class in the case where M and N are compact manifolds with $K_N \leqslant 0$. They deformed an arbitrary map through a nonlinear heat equation. By passing to the limit, with the appropriate estimates, one obtains a harmonic map in this way. In fact, harmonic maps are unique in their homotopy classes if $K_N < 0$ and rank $\geqslant 2$ [Hr]. Later, R. Hamilton [Ha] using the same method as in [ES] together with delicate estimates, settled the Dirichlet problem when M

is a manifold with boundary. This type of argument breaks down when we drop the non-positivity condition. For example Eells and Wood [EW1] have shown that there does not exist a degree 1 map from a 2-torus to a 2-sphere.

Instead of looking for harmonic maps in a homotopy class, one can look for harmonic maps with the same action on π_1. We say that two maps $f, g: M \to N$ are π_1- equivalent if $f_* = g_*: \pi_1(M) \to \pi_1(N)$. When M is a Riemann surface, L. Lemaire [Lm] proved the existence of a regular, energy minimizing harmonic map in the class of π_1-equivalent maps.

Another treatment of this problem was given by Sacks-Uhlenbeck [SaU] and R. Schoen-S. T. Yau [Sc-Y1]. Schoen-Yau considered the function space L_1^2 and showed that for $u \in L_1^2(M, N)$, u_* is well-defined and preserved under the weak limit. Using the class $\{f \in L_1^2(M, N) \mid f_* = (f_0)_*\}$ which is weakly closed, combined with the regularity of minimizing harmonic maps from a surface, one can show the existence of a smooth harmonic map in this class.

Schoen-Yau's argument could be generalized to higher dimensions by restricting the map f to the two skeleton of M. (This was also observed by White [Wh].) It is reasonable to expect that one can produce an energy minimizing harmonic map whose action on $\pi_2(M)$ has some resemblance to a given map.

For minimizing harmonic maps, R. Schoen and K. Uhlenbeck [ScU1, 2] have done fundamental work. By delicate use of comparison maps, they showed that the Hausdorff dimension of the singular set of energy minimizing harmonic maps is of codimension at least three. Their theorem can be used to recover the former theorems of Eells-Sampson and Sachs-Uhlenbeck.

2. Noncompact manifolds

The theory for harmonic maps between noncompact manifolds is more complicated than when the manifolds are compact. One reason is that when we choose a minimizing sequence of maps, their energies may not be concentrated in a bounded region. On the other hand, one hopes that this can be prevented by making suitable topological assumptions on the manifolds.

For L^2-harmonic maps, i.e., weakly harmonic maps with finite energy, one can sometimes prove existence by making geometric or topological restrictions. When N is a manifold with nonpositive curvature, Schoen and