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AN ORBIT CLOSING PROOF OF BROUWER'S LEMMA ON TRANSLATION ARCS

by Albert FATHI [1])

ABSTRACT

We give a proof of the part of Brouwer's plane translation theorem which says that an orientation preserving homeomorphism of the plane with no fixed point cannot have a non wandering point.

There are several proofs of Brouwer's lemma on translation arcs-the cornerstone of his theorem on plane translation [Bu]. A recent one was given by Morton Brown [Bw1]. The proof given here is a better restatement of a proof I gave in a graduate course lecture in December 1980 in Orsay. The improvement is largely due to the interest shown by John Franks during our stay in Warwick in June 1986 for the special year on Smooth Ergodic Theory. In fact, John Franks' work on the Poincaré-Birkhoff theorem, see [F1] and [F2], contains a very nice use of Brouwer's lemma. One of the two main tricks used below is to use a global form of closing an orbit, the use of the local form of closing orbit is also one of the main ideas of [F1] and [F2]. I would like also to thank Lucien Guillou who introduced me to Brouwer's lemma and provided me with the first proof I ever understood of this lemma.

We recall a couple of definitions and notations. If $h: Z \to Z$ is a homeomorphism of the topological space Z, we denote by $\mathrm{Fix}(h)$ the set of fixed points of h and by $\mathrm{supp}(h)$ its support which is the closure in Z of the set $\{z \in Z \mid h(z) \neq z\}$.

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1. An isotopy closing lemma

We will prove the following closing lemma:

Main Lemma 1.1. *Let* $h: M \to M$ *be a homeomorphism of the connected manifold* M. *If* h *has a nonwandering point which is not a fixed point, then there exists an isotopy* $\{h_t \mid t \in [0, 1]\}$ *such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \backslash \mathrm{Fix}(h)$ *which does not depend on* t;

(iii) $\mathrm{Fix}(h_t) = \mathrm{Fix}(h)$;

(iv) h_1 *has a periodic point of period 2 in* $M \backslash \mathrm{Fix}(h_1)$.

We will need several elementary lemmas. The first lemma is easy.

Lemma 1.2. *Let* $\varphi_1, ..., \varphi_k$ *be homeomorphisms of a space* Z. *If* X *is a subset of* Z, *we have*

$$\varphi_k \dots \varphi_1(X) \subset X \cup \left(\bigcup_{i=1}^{k} \mathrm{supp}(\varphi_i)\right).$$

Lemma 1.3. *Suppose that* h *and* $\varphi_1, ..., \varphi_k$ *are homeomorphisms of the space* Y. *If we have*

$$\forall i = 1, ..., k, h(\mathrm{supp}\, \varphi_i) \cap \left(\bigcup_{j \leqslant i} \mathrm{supp}\, \varphi_j\right) = \emptyset$$

then $\mathrm{Fix}(\varphi_k \dots \varphi_1 h) = \mathrm{Fix}(h)$.

Proof. Since $h(\mathrm{supp}\, \varphi_i) \cap \mathrm{supp}\, \varphi_i = \emptyset$, we have

$$\mathrm{Fix}(h) \cap \left(\bigcup_{i=1}^{k} \mathrm{supp}\, \varphi_i\right) = \emptyset .$$

This implies the inclusion $\mathrm{Fix}(h) \subset \mathrm{Fix}(\varphi_k \dots \varphi_1 h)$.

We prove the other inclusion by induction on k.

Suppose $k = 1$. If $\varphi_1 h(x) = x$ and $h(x) \neq x$ then certainly $h(x) \in \mathrm{supp}\, \varphi_1$ and hence also $x = \varphi_1 h(x) \in \mathrm{supp}\, \varphi_1$. But this is impossible, since $h(\mathrm{supp}\, \varphi_1) \cap \mathrm{supp}\, \varphi_1 = \emptyset$.

Suppose the lemma true for $k - 1$. Let x be such that $\varphi_k \varphi_{k-1} \dots \varphi_1 h(x) = x$. This is equivalent to $\varphi_{k-1} \dots \varphi_1 h(x) = \varphi_k^{-1}(x)$. If $x \notin \mathrm{supp}\, \varphi_k$, we obtain $\varphi_{k-1} \dots \varphi_1 h(x) = x$. By the induction hypothesis, this gives $x \in \mathrm{Fix}(h)$. If

$x \in \operatorname{supp} \varphi_k$, then $h(x) \in h(\operatorname{supp} \varphi_k)$, and since $h(x) = \varphi_1^{-1} \dots \varphi_{k-1}^{-1} \varphi_k^{-1}(x)$, we obtain, by 1.2, that $h(x) \in \bigcup_{j \leqslant k} \operatorname{supp} \varphi_j$ which is disjoint from $h(\operatorname{supp} \varphi_k)$! $\square$

The next definition is due to Brouwer.

Definition 1.4 (*Translation arc*). Let $h: Z \to Z$ be a homeomorphism of the space Z. An injective arc $\alpha \subset Z$ is called a translation arc (for h) if α joins some point x to its image $h(x)$ and $h(\alpha) \cap \dot{\alpha} = \emptyset$, where $\dot{\alpha}$ is α minus its extremities. Remark that α does not contain any of the fixed points of h. Moreover, we have $h(x) \in \alpha \cap h(\alpha)$ and if $\alpha \cap h(\alpha) \neq \{h(x)\}$ then $x = h^2(x)$.

LEMMA 1.5 (Brouwer). *Let $h: M \to M$ be a homeomorphism of the manifold M. If y and $h(y)$ are contained in the same component of* $M \backslash \operatorname{Fix}(h)$, *then there exists a translation arc α with $y \in \dot{\alpha}$.*

Proof (Well known). We can assume M connected and $\operatorname{Fix}(h) = \emptyset$. Let B be a subset of M homeomorphic to the euclidean closed ball of the same dimension as M, containing y in its interior and with $h(B) \cap B = \emptyset$. Since M is connected, there exists an isotopy $\{\theta_t \mid t \in [0, 1]\}$ such that $\theta_0 = \mathrm{Id}$, $\theta_t(y) = y$ and $\theta_1(h(y)) \in B$. If we put $B_t = \theta_t^{-1}(B)$, there is a first t such that $B_t \cap h(B_t) \neq \emptyset$, we call s this first t. We have:

(i) y is in the interior of B_s;

(ii) the interiors of B_s and $h(B_s)$ are disjoint;

(iii) B_s intersects $h(B_s)$ in a point which on the boundary of each one of them. If we call $h(x)$ this point, then x is also in the boundary of B_s.

It follows that we can find an arc $\alpha \subset B_s$ between x and $h(x)$, with $\dot{\alpha}$ contained in the interior of B_s. By (ii) above, $h(\alpha) \cap \dot{\alpha} = \emptyset$. $\square$

PROPOSITION 1.6. *Let α be a translation arc for the homeomorphism h of the connected manifold M. If for some $n \geqslant 2$ we have $h^n(\alpha) \cap \alpha \neq \emptyset$, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \backslash \operatorname{Fix}(h)$ *which does not depend on* t;

(iii) $\operatorname{Fix}(h_t) = \operatorname{Fix}(h)$;

(iv) h_1 *has a periodic point of period 2 in* $M \backslash \operatorname{Fix}(h_1)$.

Proof. We call $x, h(x)$ the extremities of α. By 1.4, we are reduced to the case $\alpha \cap h(\alpha) = \{h(x)\}$. Call $n+1$ the first integer $\geqslant 2$ such that $h^{n+1}(\alpha) \cap \alpha \neq \emptyset$. Let $z \in h^{n+1}(\alpha) \cap \alpha$. By our choice of $n+1$, if $n+1 \geqslant 3$ and the fact that $\alpha \cap h(\alpha) = \{h(x)\}$, if $n+1 = 2$, we have $z \neq h(x)$. We orient the injective segment $\bigcup_{i=0}^{n} h^i(\alpha)$ from x to $h^{n+1}(x)$. We denote by $\leqslant$ the natural order induced by this orientation with $x < h(x)$. We first consider the case where $h^{-2}(z) \leqslant z$. Let $\beta \subset \alpha \backslash \{h(x)\}$ be the compact sub arc joining $h^{-2}(z)$ to z. We have $h(\beta) \cap \beta = \emptyset$. Let V be a small connected neighborhood of β such that $h(V) \cap V = \emptyset$. Call φ_t an isotopy of M with compact support contained in V and such that $\varphi_0 = \mathrm{Id}$ and $\varphi_1(z) = h^{-2}(z)$. We can define h_t as $\varphi_t h$. By 1.3, the conditions $h(V) \cap V = \emptyset$ and $\mathrm{supp}(\varphi_t) \subset V$ imply that $\mathrm{Fix}(\varphi_t h) = \mathrm{Fix}(h)$. Furthermore, since $h^{-2}(z) \in V$, we have $h^{-1}(z) \in h(V)$ which does not intersect $\mathrm{supp}(\varphi_1)$. It follows that $(\varphi_1 h)(h^{-2}(z)) = h^{-1}(z)$, and hence we obtain $(\varphi_1 h)^2(h^{-2}(z)) = \varphi_1(z) = h^{-2}(z)$.

We now consider the case $z \leqslant h^{-2}(z)$. We choose $z_0 = z \leqslant z_1 \leqslant \dots \leqslant z_k = h^{-2}(z)$ in the segment $\bigcup_{i=0}^{n-1} h^i(\alpha)$ such that the subsegment $[z_0, z_i]$ is disjoint from the image $h([z_{i-1}, z_i])$, for $i = 1, \dots, k$. We can find neighborhoods $V_1, \dots, V_i, \dots, V_k$ of $[z_0, z_1], \dots, [z_{i-1}, z_i], \dots, [z_{k-1}, z_k]$ such that $h(V_i) \cap (\bigcup_{j \leqslant i} V_j) = \emptyset$. It is easy to construct a sequence of isotopies with compact support $\varphi_t^1, \dots, \varphi_t^k$ such that $\varphi_1^i(z_{i-1}) = z_i$ and $\mathrm{supp}\, \varphi_t^i \subset V_i$. By 1.3, this last condition and the fact that $h(V_i) \cap (\bigcup_{j \leqslant i} V_j) = \emptyset$, for $i = 1, \dots, k$, imply the equality $\mathrm{Fix}(\varphi_t^k \dots \varphi_t^1 h) = \mathrm{Fix}(h)$. Moreover, since $h^{-1}(z) \in h(V_k)$ which is disjoint from $\bigcup_{i=1}^{k} \mathrm{supp}\, \varphi_i$, we have $(\varphi_1^k \dots \varphi_1^1 h)^2(h^{-2}(z)) = h^{-2}(z)$. □

COROLLARY 1.7. *Let α be a translation arc for the homeomorphism h of the connected manifold M. Suppose that some point of α is in the closure of $\bigcup_{n \geqslant 2} h^n(\alpha)$, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \backslash \mathrm{Fix}(h)$;

(iii) $\mathrm{Fix}(h_t) = \mathrm{Fix}(h)$;

(iv) h_1 *has a periodic point of period 2 in* $M \backslash \mathrm{Fix}(h_1)$.

Proof. We can suppose that $\alpha \cap (\bigcup_{n \geqslant 2} h^n(\alpha)) = \emptyset$. Then we will find an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:

(i) $h_0 = h$;

(ii) α is a translation arc for each h_t;

(iii) $\mathrm{Fix}(h_t) = \mathrm{Fix}(h)$;

(iv) $h_1^n(\alpha) \cap \alpha \neq \emptyset$, for some $n \geqslant 2$;

(v) $h_t = h$ outside a compact subset of M which does not depend on t.

It will then suffice to apply proposition 1.6 to h_1.

We denote by x and $h(x)$ the extremities of α. Let us call $z \in \alpha \backslash \{h(x)\}$ a point of accumulation of $\bigcup_{n \geqslant 2} h^n(\alpha)$. Let V be a small connected neighborhood of z which does not intersect $h(\alpha)$. Let $n \geqslant 2$ be the first integer such that $h^n(\alpha)$ intersects V. We can find an isotopy $\{\varphi_t \mid t \in [0, 1]\}$, with compact support contained in V, such that $\varphi_0 = \mathrm{Id}$ and $\varphi_1 h^n(\alpha) \ni z$. It suffices to define h_t as $\varphi_t h$. □

Lemma 1.8. *Let h be a homeomorphism of the manifold M. Suppose that h has a non-wandering point for h which is not a fixed point, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:*

(i) $h_0 = h$;

(ii) $h_t = h$ *outside a compact subset of* $M \backslash \mathrm{Fix}(h)$;

(iii) $\mathrm{Fix}(h_t) = \mathrm{Fix}(h)$;

(iv) *there is a periodic point of* h_1 *which is not a fixed point.*

Proof. Call z a non-wandering point which is not a fixed point. Let V be a small open connected neighborhood of z such that $h(V) \cap V = \emptyset$. Call $n \geqslant 2$ the first integer such that $h^n(V) \cap V \neq \emptyset$. Choose $y \in h^{-n}(V) \cap V \neq \emptyset$. Call $\{\varphi_t \mid t \in [0, 1]\}$ an isotopy with compact support in V and such that $\varphi_1(h^n(y)) = y$. It suffices to put $h_t = \varphi_t h$. □

Proof of the Main Lemma. If h leaves invariant each component of $M \backslash \mathrm{Fix}(h)$, the Main Lemma follows from what we have done. If this is not the case then by a result of Brown and Kister [BK] $M \backslash \mathrm{Fix}(h)$ has exactly two connected components which are exchanged by h. It is easy to construct the required isotopy in this case. □

Remarks 1.9. (i) In the proof of the Main Lemma, we use the Brown-Kister result only in the case where $\mathrm{Fix}(h)$ disconnects x from $h(x)$. In particular, if M is connected, of dimension $\geqslant 2$, and if $\mathrm{Fix}(h)$ is finite we do not have to use it.

(ii) It follows from [Bw2, Lemma 6.3] that a homeomorphism of a connected manifold of dimension $\geqslant 3$ which is not the identity can be isotoped without changing the set of fixed point to a homeomorphism with a periodic point of period 2. Hence, the main lemma 1.1 is useful only for dimension 2.

2. FIXED POINTS FOR HOMEOMORPHISMS OF THE SPHERE

The next lemma is the second important ingredient. It can be proved by Nielsen's theory of fixed points. We will give a direct proof.

LEMMA 2.1. *Let* $h: \mathbf{S}^2 \to \mathbf{S}^2$ *be an orientation preserving homeomorphism. If* h *has a period 2 point which is not a fixed point, then the set* $\mathrm{Fix}(h)$ *can be written as a disjoint union* $\mathrm{Fix}(h) = F_1 \cup F_2$ *with* F_1 *and* F_2 *closed non empty and having point index equal to 1.*

Proof. Call x the point of period 2. Remark that since h preserve the orientation it induces on $\pi_1(\mathbf{S}^2 \backslash \{x, h(x)\}) = \mathbf{Z}$ the map $x \mapsto -x$. Choose an essential annulus $A \subset \mathbf{S}^2 \backslash \{x, h(x)\}$ large enough so that when we compose $h: A \to \mathbf{S}^2 \backslash \{x, h(x)\}$ with a retraction of $\mathbf{S}^2 \backslash \{x, h(x)\}$ on A we obtain a map $\bar{h}: A \to A$ which has no fixed point on the boundary, has the same fixed point as h and is equal to h in a neighborhood of the set of fixed points $\mathrm{Fix}(h) = \mathrm{Fix}(\bar{h})$. We will call $\tilde{A} \to A$ the universal cover of A of course $\tilde{A} = [0, 1] \times \mathbf{R}$ and if we denote by T a generator of the group of deck transformation of $\tilde{A} \to A$, we can write under this identification $T(x) = x + 1$ where addition is to be taken in the $\mathbf{R}$ coordinate. The map $\bar{h}$ lifts to a proper map $\tilde{h}$ which verifies $\tilde{h}T = T^{-1}\tilde{h}$. It follows that $\tilde{h}$ can be extended to the compactification of $\tilde{A}$ by its two ends $\varepsilon_-, \varepsilon_+$ by a map which exchange these to ends. Since $\tilde{A} \cup \{\varepsilon_-, \varepsilon_+\}$ is homeomorphic to a disk $\tilde{h}$ has a non empty compact set $\tilde{F}_1$ of fixed points which does not intersect the boundary because $\tilde{h}$ exchange ε_- and ε_+ and $\bar{h}$ has no fixed point on the boundary of A. Remark that the index of $\tilde{F}_1$ is 1. Moreover, the map $\tilde{A} \to A$ is injective on $\tilde{F}_1$ because if $\tilde{h}(x) = x$ we have $\tilde{h}(x+n) = x - n \neq x + n$ if $n \neq 0$. Since $\tilde{A} \to A$ is a covering it is clear that this map is also injective in a neighborhood of $\tilde{F}_1$. It follows that the image F_1 of $\tilde{F}_1$ under $\tilde{A} \to A$ is a compact non empty set of fixed points of $\bar{h}$ which has index 1. If $x \in \tilde{F}_1$, we have $T\tilde{h}(x+n) = T(x-n) = x + 1 - n \neq x + n$ for all n because $1/2 \notin \mathbf{Z}$. It follows that F_2, the image under $\tilde{A} \to A$ of $\mathrm{Fix}(T\tilde{h})$—which is also a compact non empty set of fixed points of $\bar{h}$ with index 1—is disjoint from F_1. If $x \in A$ is a fixed point of $\bar{h}$, it lifts to a point $\tilde{x} \in \tilde{A}$ which verifies $\tilde{h}(\tilde{x}) = \tilde{x} + n$. If $n = 2k$ then $\tilde{h}(\tilde{x}+k) = \tilde{h}(\tilde{x}) - k = \tilde{x} + 2k - k = \tilde{x} + k$. If $n = 2k - 1$ then $T\tilde{h}(\tilde{x}+k) = T(\tilde{x}+2k-1-k) = \tilde{x} + k$. This shows clearly that $\mathrm{Fix}(\bar{h}) = F_1 \cup F_2$. Since $\bar{h}$ is equal to h in a neighborhood of $\mathrm{Fix}(\bar{h}) = \mathrm{Fix}(h)$, this ends the proof. □

If we combine the Main Lemma 1.1 and lemma 2.1, we obtain:

LEMMA 2.2. *Let* $h: \mathbf{S}^2 \to \mathbf{S}^2$ *be an orientation preserving homeomorphism. If* h *has a non wandering point which is not a fixed point, then the set* $\mathrm{Fix}(h)$ *can be written as a disjoint union* $\mathrm{Fix}(h) = F_1 \cup F_2$ *with* F_1 *and* F_2 *closed non empty and having fixed point index equal to 1.*

Since we can compactify an orientation preserving homeomorphism of $\mathbf{R}^2$ by an orientation preserving homeomorphism of $\mathbf{S}^2$ with one more fixed point at infinity, we obtain the next two corollaries.

COROLLARY 2.3. (Brouwer's Lemma on translation arcs). *Let* $h: \mathbf{R}^2 \to \mathbf{R}^2$ *be a fixed point free orientation preserving homeomorphism. Then* h *has no periodic point, each point wanders under* h. *Moreover, if* α *is a translation arc, the union* $\bigcup_{n \in \mathbf{Z}} h^n(\alpha)$ *is homeomorphic to a line and it does not accumulate on itself.*

COROLLARY 2.4. *Let* $h: \mathbf{R}^2 \to \mathbf{R}^2$ *be an orientation preserving homeomorphism. If the non wandering set of* h *is not reduced to the set of fixed points then there is a compact non empty subset* $F \subset \mathrm{Fix}(h)$ *which has fixed point index equal to 1.*

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