§0. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **32 (1986)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

L'Enseignement Mathématique, t. 32 (1986), p. 95-110

GEOMETRY AND COMBINATORICS OF GROUPS GENERATED BY REFLECTIONS

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§ 0. INTRODUCTION

Let M be a connected topological manifold. A continuous transformation s of M is called a reflection if $s^2 = 1$ and the set M_s of fixed points of s has codimension one. A reflection s is called separating if $M \backslash M_s$ is disconnected.

In this paper we consider groups $G: M \to M$ of transformations generated by separating reflections and acting properly discontinuously on M. We call them reflection groups for brevity. Reflection groups occur in various branches of mathematics. Weyl groups of semisimple Lie groups is a well known example (see [2] and the references there). Another important example of reflection groups is the Weyl groups of Kac-Moody Lie groups (cf. [13]) and of the loop groups (cf. [16]) acting properly discontinuously on the Tits cone (cf. [20]).

Reflection groups have applications in such different areas as the classification of singularities (cf. [1], [14], [17] to mention but a few), integrable quantum many body problems (cf. [5]-[7], [10] and [12]) and the minimal surfaces ([15]). Recently M. Davis [3] found interesting applications of reflection groups in topology.

In the majority of applications of reflection groups they act by linear, affine, projective or hyperbolic transformations on linear, affine, projective or hyperbolic spaces respectively. But the most useful properties of reflection groups are independent of the particular nature of the group action. The properties I have in mind are combinatorial or geometric (cf. Theorem 1 and

[1]) Research partially supported by NSF Grant DMS 84-03238.

Corollary 5 in § 1) which hold for reflection groups in general as was demonstrated by Davis [3] (see also [18]).

The proofs of these general properties in the literature depend on the particular type of the reflection groups considered. The best known proofs are those in [2] for the linear (or affine) reflection groups acting properly discontinuously on the whole space, and Davis [3] adapted them to the general case of topological reflection groups.

The purpose of this paper is to supply geometric proofs of the basic properties of general reflection groups as opposed to adapting the formal arguments of [2]. For simplicity of exposition we assume in the paper that M is a differentiable (actually C^1) manifold and that the group action is C^1. Extension to the topological manifolds does not require new ideas and is left to the reader.

The rationale for this paper is twofold. First, the basic properties of reflection groups are stated (and proved) here in a form particularly useful for applications (cf. [5], [9], [11]). Second and more important, the simplicity of the geometric proofs presented here will make the subject more accessible to the general mathematical public.

In conclusion let me mention that reflection groups that do not act properly discontinuously are also useful (cf. [19], [4], [8]) but the results of the paper do not extend to them.

I would like to thank Mike Davis and the referee for pointing out an error in the original version of the paper. [1])

§ 1. Geometry and combinatorics

Throughout the paper M is a connected differentiable manifold (possibly with boundary).

Definition 1. A reflection of M is a diffeomorphism s such that $s^2 = 1$ and the set M_s of fixed points of s has codimension 1. A reflection s is called separating if $M \backslash M_s$ is disconnected. A reflection group W acting on M is a discrete group of diffeomorphisms of M generated by separating reflections.

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[1]) I. N. Bernstein told me that E. B. Vinberg has an unpublished manuscript on reflection groups which is similar to this one.