

# 1. Graphs

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## TREES, TAIL WAGGING AND GROUP PRESENTATIONS

by M. A. ARMSTRONG

The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

### 1. GRAPHS

A graph  $X$  consists of two sets  $E$  (directed edges) and  $V$  (vertices) and two functions

$$\begin{aligned} E &\rightarrow E, & e &\mapsto \bar{e} \\ E &\rightarrow V \times V, & e &\mapsto (i(e), t(e)) \end{aligned}$$

which satisfy  $\bar{\bar{e}} = e$ ,  $\bar{e} \neq e$  and  $i(\bar{e}) = t(e)$  for each  $e \in E$ . The vertices  $i(e)$ ,  $t(e)$  are the initial and terminal vertices of the directed edge  $e$ , and  $\bar{e}$  is the reverse of  $e$ . Henceforth we refer to directed edges simply as edges.

A path in  $X$  joining vertex  $u$  to vertex  $v$  is an ordered string of edges  $e_1 e_2 \dots e_n$  such that  $i(e_1) = u$ ,  $i(e_{k+1}) = t(e_k)$  for  $1 \leq k \leq n-1$ , and  $t(e_n) = v$ . If  $v = u$  we have a circuit. A path of the form  $e\bar{e}$  is a *round trip* and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A *tree* is a connected graph which does not contain any loops.

Let  $X$  be a tree. A path in  $X$  is a *geodesic* if it does not contain any round trips. Given distinct vertices  $u, v$  of  $X$  there is a *unique* geodesic  $\overrightarrow{uv}$  which joins  $u$  to  $v$ .

An action of a group  $G$  on a graph  $X$  is an action of  $G$  on  $E$  and on  $V$  such that  $g\bar{e} = \overline{ge}$ ,  $i(ge) = gi(e)$ ,  $t(ge) = gt(e)$  and  $ge \neq \bar{e}$  for each  $e \in E$ . Because group elements are not allowed to reverse edges we have a

quotient graph  $X/G$ . When  $G$  acts on  $X$  we shall often say that  $G$  is a *group of automorphisms of  $X$* .

We adopt the usual notation whereby  $G_x$  denotes the stabilizer of a vertex  $x$ . If  $g \in G$  happens to fix  $x$  we write  $g_x$  for the element  $g$  thought of as a member of  $G_x$ . Of course  $G_e$  denotes the stabilizer of the edge  $e$ . If  $x$  is a vertex of  $e$  then  $G_e$  is a subgroup of  $G_x$ .

Suppose  $G$  acts on a tree  $X$ . If  $g \in G$  fixes the vertices  $u, v$  then it must fix the whole geodesic  $\overrightarrow{uv}$ , since otherwise the image of  $\overrightarrow{uv}$  under  $g$  would be a second geodesic from  $u$  to  $v$ .

## 2. LIFTING EDGES

Let  $G$  be a group of automorphisms of a tree  $X$ . Choose a maximal tree  $M$  in  $X/G$  and lift it [4, Proposition I.14] to a subtree  $T$  of  $X$ . The vertices of  $T$  form a set of representatives for the action of  $G$  on the vertices of  $X$ . For each pair of edges  $f, \bar{f}$  from  $X/G - M$  select one, say  $f$ , and lift it to an edge  $e$  of  $X$  which has its initial vertex  $x$  in  $T$ . Exactly one vertex  $z$  of  $T$  lies in the same orbit as  $t(e)$  and we choose an element  $\gamma_f$  from  $G$  that maps  $z$  onto  $t(e)$ . We can now lift  $\bar{f}$  to  $(\gamma_f)^{-1}\bar{e}$ . This has its initial vertex  $z$  in  $T$  and  $\gamma_{\bar{f}} = (\gamma_f)^{-1}$  sends the vertex  $x$  of  $T$  to its terminal vertex (Figure 1). Finally we extend the correspondence  $f \rightarrow \gamma_f$  over the edges of  $M$  by setting  $\gamma_f = 1$  (the identity element of  $G$ ) whenever  $f \in M$ .

The *Bass-Serre theorem* [4, Theorem I.13] gives the following presentation for  $G$ .

(a) *Generators.* The elements of all the  $G_w$  where  $w$  is a vertex of  $T$  and the  $\gamma_f$  where  $f$  is an edge of  $X/G$ .

(b) *Relations.* The internal relations of each stabilizer  $G_w$  together with

$$\gamma_f = 1 \text{ if } f \text{ is an edge of } M,$$

$$\gamma_{\bar{f}} = (\gamma_f)^{-1} \text{ and}$$

$$\gamma_{\bar{f}} g_x \gamma_f = (\gamma_{\bar{f}} g \gamma_f)_z \text{ where } e \text{ is the chosen lift of } f \text{ and } g \in G_e.$$

(If  $f$  is an edge of  $M$  then  $z = t(e)$  and the final relation reduces to  $g_x = g_z$  whenever  $g \in G_e$ ).