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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **32 (1986)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-55088

L'Enseignement Mathématique, t. 32 (1986), p. 219-226

ON CONSECUTIVE VALUES OF THE LIOUVILLE FUNCTION

by Adolf HILDEBRAND

ABSTRACT: It is shown that for every choice of $\varepsilon_i = \pm 1, i = 1, 2, 3$, there exist infinitely many positive integers n, such that $\lambda(n+i) = \varepsilon_i$, $i = 1, 2, 3$, where λ denotes the Liouville function. [1])

1. INTRODUCTION

Let $\lambda(n)$ denote the Liouville function, i.e. $\lambda(n) = +1$ or -1 according as the number of prime factors of n (counted with multiplicity) is even or odd. It is natural to expect that the sequence $(\lambda(n))$ behaves like a random sequence of $\pm$ signs. A particularly attractive and highly plausible conjecture is that every finite "block" of $\pm$ signs occurs in this sequence infinitely often, i.e. for any given numbers $\varepsilon_i = \pm 1, 1 \leqslant i \leqslant k$, there are infinitely many integers $n \geqslant 1$, such that

$$\lambda(n+i) = \varepsilon_i \quad (1 \leqslant i \leqslant k).$$

Whereas for $k = 1$ and $k = 2$ this conjecture holds trivially, there are no results known in the literature for larger values of k. In [1, p. 95, problem 56], Chowla states the above conjecture and remarks that "for $k \geqslant 3$, this seems an extremely hard conjecture". The purpose of this paper is to prove the conjecture in the first non-trivial case $k = 3$.

THEOREM. *For any choice of* $\varepsilon_i = \pm 1, i = 1, 2, 3$, *there are infinitely many positive integers* n *such that*

$$\lambda(n+i) = \varepsilon_i \quad (i=1, 2, 3). \tag{1}$$

We shall use for the proof an "ad hoc" method, which leads in a relatively simple way and using only very elementary arguments to the

[1]) *1980 A.M.S. Subject Classification:* Primary 10 H 25, Secondary 10 K 20, 10 A 20.

desired result. The drawback of this method is that it gives no indication on how to settle the general case of the conjecture, or even the case $k = 4$. It seems that for this completely new ideas are needed, and Chowla's remark on the difficulty of the problem appears to be justified, as far as the general form of the conjecture is concerned.

2. A Lemma

Lemma. *Each of the equations*

$$\lambda(15n-1) = \lambda(15n+1) = 1$$

and

$$\lambda(15n-1) = \lambda(15n+1) = -1$$

holds for infinitely many positive integers n.

Proof. Given a positive integer $n_0 \geqslant 2$, define n_i, $i \geqslant 1$, inductively by

$$n_{i+1} = n_i(4n_i^2-3) \quad (i \geqslant 0).$$

It is easily checked that

$$n_{i+1} \pm 1 = (n_i \pm 1)(2n_i \mp 1)^2 \quad (i \geqslant 0),$$

so that

$$\lambda(n_{i+1} \pm 1) = \lambda(n_i \pm 1) = \dots = \lambda(n_0 \pm 1) \quad (i \geqslant 0).$$

Also, it follows by induction that $n_0 \mid n_i$ for all $i \geqslant 0$. Therefore, taking in turn $n_0 = 15$ and $n_0 = 30$ and noting that

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1,$$

we obtain two infinite sequences $(n_i^{(+)})$ and $(n_i^{(-)})$ with the required properties

$$n_i^{(\pm)} \equiv 0 \pmod{15}, \quad \lambda(n_i^{(+)} \pm 1) = 1, \quad \lambda(n_i^{(-)} \pm 1) = -1.$$

Remark. The same argument shows that for any completely multiplicative function f assuming only the values ± 1 and for given $\varepsilon_1, \varepsilon_2 = \pm 1$ and $a \geqslant 2$, the system

$$n \equiv 0 \pmod{a}, \quad f(n-1) = \varepsilon_1, \quad f(n+1) = \varepsilon_2$$

has infinitely many solutions, provided it has at least one solution. It would be interesting to have an analogous result for three (or more) consecutive values, but the above method does not work in this case.

3. Proof of the Theorem, beginning

We shall show here that each of the equations

$$\lambda(n) = \lambda(n+1) = \lambda(n-1) = 1 \tag{2}$$

and

$$\lambda(n) = \lambda(n+1) = \lambda(n-1) = -1 \tag{2'}$$

has infinitely many solutions. Since the arguments for the two cases are completely symmetric, we shall carry out the proof only in the case of equation (2).

Call an integer $n \geqslant 2$ "good", if (2) holds for this n. We have to show that there are infinitely many good integers. To this end we shall show that for any positive integer n satisfying

$$n \equiv 0 \pmod{15}, \quad \lambda(n+1) = \lambda(n-1) = 1, \tag{3}$$

the interval

$$I_n = \left[\frac{4n}{5}, 4n + 5\right] \tag{4}$$

contains a good integer. Since by the lemma (3) holds for infinitely many positive integers n, the desired result follows.

To prove our assertion we fix a positive integer n, for which (3) holds. We may suppose $\lambda(n) = -1$, since otherwise $n \in I_n$ is good, and we are done. Put $N = 4n$, and note that, by construction, N is divisible by 3, 4 and 5. From (3) we get, using the multiplicativity of the function λ,

$$\lambda(N \pm 4) = \lambda\big(4(n \pm 1)\big) = \lambda(4)\,\lambda(n \pm 1) = 1\,,$$

and our assumption $\lambda(n) = -1$ implies

$$\lambda(N) = \lambda(4n) = \lambda(4)\,\lambda(n) = -1\,.$$

If now

$$\lambda(N+5) = \lambda(N-5) = -1\,,$$

then

$$\lambda\left(\frac{N}{5} \pm 1\right) = \frac{\lambda(N \pm 5)}{\lambda(5)} = 1 = -\lambda(N) = \lambda\left(\frac{N}{5}\right),$$

and $N/5 = 4n/5 \in I_n$ is good. We may therefore suppose that at least one of values $\lambda(N+5)$ and $\lambda(N-5)$ equals 1.

For definiteness we shall assume $\lambda(N+5) = 1$; the other case is treated in exactly the same way.

If $\lambda(N+3) = 1$ or $\lambda(N+6) = 1$, then $N + 4 \in I_n$ or $N + 5 \in I_n$ is good. But in the remaining case

$$\lambda(N+3) = \lambda(N+6) = -1$$

we have

$$\lambda\left(\frac{N}{3}\right) = \lambda\left(\frac{N}{3} + 1\right) = \lambda\left(\frac{N}{3} + 2\right) = 1,$$

so that $(N+3)/3 \in I_n$ is good.

Thus (3) implies the existence of a good integer in the interval (4), as we had to show.

4. Proof of the Theorem, conclusion

So far we have proved that (1) has infinitely many solutions in the cases $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$. But this obviously implies that for each of the triples $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, -1), (-1, -1, 1), (1, -1, -1)$ and $(-1, 1, 1)$ there are also infinitely many solutions to (1). It remains therefore to consider the triples $(1, -1, 1)$ and $(-1, 1, -1)$. Since the arguments in both cases are the same (with $+1$ and -1 interchanged), we shall confine ourselves to the case $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$. Accordingly, we call $n \geqslant 2$ good, whenever

$$\lambda(n+1) = \lambda(n-1) = 1, \quad \lambda(n) = -1.$$

We have to show that there are infinitely many such n.

Suppose, to get a contradiction, that there are only finitely many good integers, all of them $\leqslant N_0$, say. Suppose further that

$$\lambda(n) = 1 (m_0 \leqslant n \leqslant n_0) \tag{5}$$

holds for some integers $n_0 > m_0 \geqslant 2N_0$. We shall show that then

(6) $$\lambda(n) = 1(m_i \leqslant n \leqslant n_i)$$

holds for all $i \geqslant 0$, where m_i and n_i are defined inductively by

(7) $$m_{i+1} = \left[\frac{3m_i + 1}{2}\right], \quad n_{i+1} = \left[\frac{3n_i}{2}\right] (i \geqslant 0).$$

This will easily lead to the desired contradiction.

By our assumption (5), (6) holds for $i = 0$. Assume now that (6) does not hold for all $i \geqslant 0$, and let $i \geqslant 0$ be minimal such that (6) holds for i and fails for $i + 1$. Thus, for some $n \in [m_{i+1}, n_{i+1}]$, which we shall fix, we have $\lambda(n) = -1$. Write

(8) $$2n = 3n' + \theta(\theta \in \{0, 1, -1\}).$$

From (7) we get

$$3m_i \leqslant 2m_{i+1} \leqslant 2n \leqslant 2n_{i+1} \leqslant 3n_i,$$

so that

$$m_i \leqslant n' \leqslant n_i,$$

and hence by (6) (which we assumed to hold for i)

$$\lambda(3n') = -\lambda(n') = -1.$$

Since, by our assumption $\lambda(n) = -1$,

$$\lambda(2n) = -\lambda(n) = 1,$$

we cannot have $\theta = 0$ in (8). The arguments in the cases $\theta = \pm 1$ being identical, we shall henceforth assume that (8) holds with $\theta = 1$.

We must have

$$\lambda(2(n-1)) = \lambda(3n'-1) = -1,$$

since otherwise $3n'$ would be good and

$$3n' \geqslant 3m_i \geqslant 3m_0 > N_0,$$

in contradiction to our assumption. Also, since

$$m_i \leqslant n' + 1 = \frac{2}{3}(n+1) \leqslant \left[\frac{2}{3}(n_{i+1}+1)\right] = \left[\frac{2}{3}\left(\left[\frac{3n_i}{2}\right] + 1\right)\right] \leqslant n_i,$$

we have by (6)

$$\lambda(2(n+1)) = \lambda(3(n'+1)) = -\lambda(n'+1) = -1.$$

These two identities imply

$$\lambda(n \pm 1) = -\lambda(2(n \pm 1)) = 1,$$

and since $\lambda(n) = -1$, we conclude that $n(>N_0)$ is good and therefore arrive at a contradiction.

Thus (5) (with $n_0 > m_0 \geqslant 2N_0$) implies (6) for all $i \geqslant 0$. To derive from this the desired contradiction, we suppose first that (5) holds for some $n_0 > m_0 \geqslant 2N_0$ satisfying

$$n_0 - m_0 \geqslant 3. \tag{9}$$

In other words, we suppose (for the moment) that there exist four consecutive integers $n \geqslant 2N_0$, for which $\lambda(n) = 1$. Putting $d_i = n_i - m_i$, we have, by the recursion formulae (7),

$$d_{i+1} \geqslant \frac{3}{2} d_i - 1 = \frac{3}{2} d_i \left(1 - \frac{2}{3d_i}\right) \quad (i \geqslant 0).$$

Taking into account (9), we obtain by induction in turn

$$d_i \geqslant 3 \quad (i \geqslant 0),$$

$$d_i \geqslant 3\left(\frac{7}{6}\right)^i \quad (i \geqslant 0),$$

and finally

$$d_i \geqslant \left(\frac{3}{2}\right)^i \prod_{j=0}^{i} \left(1 - \frac{2}{3d_j}\right) \geqslant C\left(\frac{3}{2}\right)^i \quad (i \geqslant 0),$$

where

$$C = \prod_{j \geqslant 0} \left(1 - \frac{2}{9}\left(\frac{6}{7}\right)^j\right) > 0.$$

Since on the other hand by (7)

$$d_i \leqslant n_i \leqslant \left(\frac{3}{2}\right)^i n_0 \quad (i \geqslant 0),$$

we see from (6), that there are arbitrary large values of x, such that $\lambda(n)$ is constant in the interval $[x(1-\varepsilon), x]$, where $\varepsilon = C/n_0$. But this is impossible since, for x sufficiently large, every such interval contains integers n and n' of the form

$$n = 4^a 9^b, \quad n' = 2 \cdot 4^c 9^d (a, b, c, d \in \mathbf{N}),$$

for which

$$\lambda(n) = 1, \quad \lambda(n') = -1.$$

We therefore have obtained the desired contradiction under the assumption that there exist four consecutive integers $n \geqslant 2N_0$, for which $\lambda(n) = 1$. By the part of the theorem already proved, there exist at least three such integers. Therefore (5) holds for some $m_0 > 2N_0$ with $n_0 = m_0 + 2$, and we may now assume that

$$\lambda(m_0 - 1) = \lambda(m_0 + 3) = -1.$$

If m_0 is odd, then this implies

$$\lambda\left(\frac{m_0 - 1}{2}\right) = \lambda\left(\frac{m_0 + 3}{2}\right) = 1, \quad \lambda\left(\frac{m_0 + 1}{2}\right) = -1,$$

so that $(m_0 + 1)/2 > N_0$ is good, in contradiction to our assumption. But if m_0 is even, then defining m_1 and n_1 by (7), (6) holds for $i = 1$, and we have

$$m_1 \geqslant 2N_0, n_1 - m_1 = \frac{3(m_0 + 2)}{2} - \frac{3m_0}{2} = 3.$$

Thus we are back in the case already treated.

By contradiction, we therefore conclude that (1) has infinitely many solutions for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$, and the proof of the theorem is complete.

5. Concluding remarks

In the foregoing proof, the relevant property of the Liouville function was that $\lambda(n)$ is completely multiplicative and assumes only the values ± 1. Besides this, we used only the fact that $\lambda(2) = \lambda(3) = \lambda(5) = -1$ and (in the proof of the lemma)

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1.$$

The proof, as it stands, works for any completely multiplicative function $f(n) = \pm 1$ with these properties. By suitably modifying the proof, it is possible to cover other classes of multiplicative functions as well.

It would be interesting to determine those completely multiplicative functions $f(n) = \pm 1$, for which the analogue of the theorem does not hold. Schur [3] proved that if $f \not\equiv f_{\pm}$, where

$$f_{\pm}(n) = \begin{cases} (\pm 1)^k & \text{if} \quad n = 3^k m, m \equiv 1 \bmod 3, \\ -(\pm 1)^k & \text{if} \quad n = 3^k m, m \equiv 2 \bmod 3, \end{cases}$$

then there exists at least one $n \geqslant 1$, such that

$$f(n) = f(n+1) = f(n+2) = 1 .$$

It is likely that under the same hypotheses there are infinitely many such n. Using arguments similar to those in section 3, one can prove this assertion under the additional hypotheses $f(2) = 1$ and $f(3) = -1$, but the general case seems to be more complicated.

A very plausible conjecture is that the integers n, for which (1) holds, have positive density. In the case $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, this would follow from an analogous strengthening of the lemma by requiring (2) to hold on a set of positive density. Whereas a very simple argument shows that the equations $\lambda(n) = \lambda(n+1)$ and $\lambda(n+1) = \lambda(n-1)$ hold on a set of positive (lower) density (cf. [2]), this argument seems to break down, if n is required to lie in a prescribed residue class, and so far we have not been able to overcome this difficulty.

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(Reçu le 21 octobre 1985)

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