

VIII. Appendix: Extending the Main Theorem to the class E OF NORM-TOPOLOGICAL SPACES

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Remark 30. By Theorem 28 the isometry type of a definite space with admissible topology is characterized by the sequence $(\langle e_i \rangle)_{i \in \mathbf{N}}$ where $(e_i)_{i \in \mathbf{N}}$ is a maximal orthogonal family in \mathfrak{E} . Conversely, for each $(\alpha_i) \in k^{\mathbf{N}}$ there is a definite space \mathfrak{E} with $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$ admitting a maximal orthogonal family $(e_i)_{i \in \mathbf{N}}$ with $\langle e_i \rangle = \alpha_i$ ($i \in \mathbf{N}$) provided that

- (A) $\xi_i := \varphi \alpha_i \in \Gamma$ satisfies the (type-) condition expressed in (8)
- (B) The form $\langle \cdot, \cdot \rangle$ defined on $\mathfrak{F} := k(e_i)_{i \in \mathbf{N}}$ by $\langle e_i, e_j \rangle = 0$ ($i \neq j$), $\langle e_i \rangle = \alpha_i$ ($i \in \mathbf{N}$) is definite.

These two conditions are implemented by many fields. In order to satisfy (A) one may, e.g. pick fields of generalized formal power series that are complete under a valuation φ with group Γ a prescribed Hahn product [30, p. 31] with sufficiently many factors not 2-divisible, e.g. $\Gamma = \mathbf{Z}^{(\mathbf{N})}$ ordered antilexicographically. Let k be any field with (A) and $t \in \Gamma/2\Gamma$; set $\mathfrak{F}_t = \{\text{span } e_i \mid \varphi \alpha_i + 2\Gamma = t\}$. By (A) $\dim \mathfrak{F}_t < \infty$; furthermore

$$\mathfrak{F} = \bigoplus^\perp \{\mathfrak{F}_t \mid t \in \Gamma/2\Gamma\}.$$

In order to check whether the form $\langle \cdot, \cdot \rangle$ satisfies the triangle inequality on \mathfrak{F} it suffices to verify said inequality on each \mathfrak{F}_t . A. Fässler has given a handy criterium for $\langle \cdot, \cdot \rangle$ to be definite if Hahnproducts Γ are used, as indicated, to construct k with (A), [6, Lemma 15, 16].

VIII. APPENDIX: EXTENDING THE MAIN THEOREM TO THE CLASS \mathcal{E} OF NORM-TOPOLOGICAL SPACES

The arguments applied to the spaces in the class \mathcal{D} can be extended to a larger class \mathcal{E} . First we have (cf. Definition 15):

Definition 31. An infinite dimensional anisotropic quadratic space $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ over a $*$ -valued field $(k, *, \varphi, \Gamma)$ is called norm-topological if the sets $\mathcal{U}_\gamma := \{x \in \mathfrak{E} \mid \varphi \langle x \rangle > \gamma\}$ form a 0-neighbourhood basis of a vector space topology on \mathfrak{E} . Let \mathcal{E} be the class of all norm-topological spaces.

Definite spaces are norm-topological, obviously.

A proper subgroup Δ of Γ is *convex* (or *isolated*) if “ $0 \leq x \leq y$ & $y \in \Delta$ ” implies “ $x \in \Delta$ ”. If the subgroup $\Delta \subset \Gamma$ is convex then the factor group Γ/Δ is ordered by setting $\gamma + \Delta \leq \delta + \Delta$ iff $\gamma < \delta$ or $\gamma - \delta \in \Delta$; furthermore, $\varphi_\Delta: k \rightarrow \Gamma/\Delta \cup \{\infty\}$ defined by $\varphi_\Delta(\alpha) = \varphi(\alpha) + \Delta$ is a valuation (a “coarser valuation”) which yields the same topology on k as φ .

In order to make the mechanism of types work in the context of norm-topological spaces, i.e., in order to salvage the statement of Corollary 26 in the new context, the concept of type has to be coarsened as follows. For $\gamma \in \Gamma$ we introduce

$$(12) \quad \Delta(\gamma) := \{ \delta \in \Gamma \mid \forall n \in \mathbf{N} : n \mid \delta \mid \leq |\gamma| \}$$

and

$$(13) \quad \Theta(\gamma) := \bigcap_{\delta \in \Gamma} \Delta(\gamma + 2\delta)$$

If $\gamma \neq 0$ then $\Delta(\gamma)$ is the largest convex subgroup of Γ not containing γ ([21]).

Remark 32. The group defined in (13) for $\gamma = \varphi\langle e \rangle$, $e \in \mathfrak{E}$, represents yet another possibility to introduce a "type" for the vectors in a definite space. The fundamental property expressed in Lemma 25 can be replaced and reproved (along the same lines), cf. [21]:

(14) If U is a convex subgroup in Γ and $(e_i)_{\mathbf{N}}$, $(f_j)_{\mathbf{N}}$ are two maximal orthogonal families in a norm-topological space that satisfies (iii) in Theorem 28 then

$$\text{card} \{ i \in I \mid \Theta(\varphi\langle e_i \rangle) \subset U \} = \text{card} \{ j \in \mathbf{N} \mid \Theta(\varphi\langle f_j \rangle) \subset U \}.$$

One has the following analogue of Lemma 14:

LEMMA 33. ([21]). *Let $(\mathfrak{E}; \langle \cdot, \cdot \rangle; \varphi, \Gamma, *)$ be a norm-topological space and $\varphi(2) = 0$ (cf. Remark 35 below). Then there is a valuation $\tilde{\varphi}: k \rightarrow \tilde{\Gamma} \cup \{\infty\}$ coarser than φ such that the following holds: Either $(\mathfrak{E}; \langle \cdot, \cdot \rangle; \tilde{\varphi}, \tilde{\Gamma}, *)$ is a definite space, in the sense of Definition 15, or else there are no analytically nilpotent elements $\alpha \in k$ (i.e., for no $\alpha \neq 0$ shall we have $\lim_{\mathbf{N}} \alpha^n = 0$) and then the following weakened versions of the statements in Lemma 14 hold:*

- (i)' $\tilde{\varphi}_{\Delta}\langle x + \eta \rangle \geq \min \{ \tilde{\varphi}_{\Delta}\langle x \rangle, \tilde{\varphi}_{\Delta}\langle \eta \rangle \}$
- (ii)' $\tilde{\varphi}\langle x \rangle \leq \tilde{\varphi}\langle \eta \rangle \ \& \ \langle x, \eta \rangle = 0 \Rightarrow \tilde{\varphi}_{\Lambda}\langle x \rangle = \tilde{\varphi}_{\Lambda}\langle x + \eta \rangle$
- (iii)' $\tilde{\varphi}\langle x, \eta \rangle \geq \min \{ \tilde{\varphi}_{\Delta}\langle x \rangle, \tilde{\varphi}_{\Delta}\langle \eta \rangle \}$
- (iv)' $2\tilde{\varphi}_{\Delta}\langle x, \eta \rangle \geq \tilde{\varphi}_{\Delta}\langle x \rangle + \tilde{\varphi}_{\Delta}\langle \eta \rangle$
 where $\Lambda = \Theta(\tilde{\varphi}\langle x \rangle)$ and $\Delta = \Theta(\tilde{\varphi}\langle x \rangle) \cap \Theta(\tilde{\varphi}\langle \eta \rangle)$.

The inequalities in Lemma 33 suffice to salvage all results proved previously on definite spaces; in particular we have the following strengthening of Theorem 28 (cf. Remark 35 below):

THEOREM 34 [21]. Let \mathfrak{E} be a norm-topological space in the sense of Definition 31 and assume $\varphi(2) = 0$. Then the statements (i), (ii), (iii) in Theorem 28 are equivalent.

Remark 35. In Definition 15, Lemma 33 and in Theorem 34 we stipulated that $\varphi(2) = 0$ for the valuation φ of the base field. However, it is neither necessary to assume this nor that $\text{char } k$ be different from two. As technicalities increase if 2 is not a unit for φ the general case has been banned from this elementary survey. Refer to [21].

IX. APPENDIX: ORTHOMODULAR SPACES OVER ORDERED FIELDS

A Baer order of a $*$ -field k is a subset $\Pi \subset S := \{\alpha \in k \mid \alpha = \alpha^*\}$ with $1 \in \Pi$, $0 \notin \Pi$, $\Pi + \Pi \subset \Pi$, $\forall \alpha \neq 0: \alpha\Pi\alpha^* \subset \Pi$, $-\Pi \cup \Pi = S \setminus \{0\}$. ([14]). The map $\alpha \mapsto \alpha^*\alpha =: \|\alpha\|$ has the properties of a norm and defines a topology on k ; if $*$ is continuous then k is a topological $*$ -field [14, Theorem 4.1, p. 231]. The theory of positive definite orthomodular spaces over archimedean ordered fields is settled in [9]: There are but the classical Hilbert spaces over \mathbf{R} , \mathbf{C} , \mathbf{H} . If the order is non-archimedean we shall assume that

(15) the subgroup S generated by all $\alpha^*\alpha^{-1}$ is bounded.

There is [14, Sec. 4.5, p. 234] a valuation on k that induces the norm-topology. We remark that the boundness condition on S is always satisfied for the usual orderings on commutative fields, for Prestel's semi-orderings and for all $*$ -ordered fields that are known hitherto.

A family $(e_l)_{l \in I}$ of vectors in a positive definite space $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ over an ordered $*$ -field k is said to satisfy the type condition (cf. Definition 21) iff for all $(\alpha_l)_{l \in I} \in k^I$ the following holds: if $(\langle \alpha_l e_l \rangle)_{l \in I}$ is bounded then $(\alpha_l e_l)_{l \in I}$ converges to $0 \in \mathfrak{E}$.

With this version of type condition we have

THEOREM 36. Let $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ be a positive definite space over a non-archimedean ordered $*$ -field that satisfies (15). Then the statements (i), (ii), (iii) in Theorem 28 are equivalent.