

VI. Sufficient conditions in D for $L_c = L_{\bot}$

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straight lines in E . A family $(e_l)_{l \in I}$ of vectors in $\mathfrak{E}$ is said to satisfy the *type-condition* iff for all $(\alpha_l)_{l \in I} \in k^I$ the following holds: if $(\varphi\langle\alpha_l e_l\rangle)_{l \in I}$ is bounded (below) then $(\alpha_l e_l)_{l \in I}$ converges to $0 \in E$.

COROLLARY 22. Let $\mathfrak{E}$ be as in Theorem 17. $\Gamma/2\Gamma$ is infinite. Each orthogonal family in $\mathfrak{E}$ satisfies the type-condition, equivalently, $\Gamma/2\Gamma$ satisfies (8) below. $\square$

COROLLARY 23. Let $\mathfrak{E}$ be as in Theorem 17. Then k is complete.

Proof. By Corollary 19 it suffices to show that a sequence $(\alpha_i)_{i \in \mathbb{N}}$ with limit $0 \in k$ is summable. Let $(e_i)_{i \in \mathbb{N}}$ be maximal orthogonal in $\mathfrak{E}$ with $(\varphi\langle e_i\rangle)_{i \in \mathbb{N}}$ bounded below. If $(\lambda_i)_{i \in \mathbb{N}} \in k^{\mathbb{N}}$ has $(\varphi(\lambda_i))_{i \in \mathbb{N}}$ bounded below then $(\lambda_i e_i)_{i \in \mathbb{N}}$ is summable and by continuity of $\langle \cdot, \cdot \rangle$ we obtain

$$\langle \sum_{\mathbb{N}} \lambda_i e_i, \sum_{\mathbb{N}} e_i \rangle = \sum_{\mathbb{N}} \lambda_i \langle e_i \rangle.$$

Thus, all families $(\lambda_i \langle e_i \rangle)_{i \in \mathbb{N}}$ with bounded $(\lambda_i)_{i \in \mathbb{N}}$ are summable.

Pick a strictly monotonic sequence $(n_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with $u_0 = 0$ and for all $i \in \mathbb{N}^+$ and all $m \geq n_i$: $\varphi(\alpha_m) > \varphi\langle e_i \rangle$, and set $A_i := \sum \{\alpha_j \mid n_i \leq j < n_{i+1}\}$. The family $(A_i)_{i \in \mathbb{N}}$ is summable if and only if $(\alpha_i)_{i \in \mathbb{N}}$ summable and, if the sums exist, these must be equal. If we set $\lambda_i := A_i \langle e_i \rangle^{-1}$ then, by what we have shown, the family of the $A_i = \lambda_i \langle e_i \rangle$ is summable. $\square$

COROLLARY 24. Let $\mathfrak{E}$ be as in Theorem 17. Then $\mathfrak{E}$ is complete.

Proof. Let $(x_i)_{i \in \mathbb{N}}$ be a Cauchy sequence (Corollary 19). For each fixed $\eta \in \mathfrak{E}$ the map $x \mapsto \langle \eta, x \rangle$ is uniformly continuous. Hence by Cor. 23 the map $f: \eta \mapsto \lim_i \langle \eta, x_i \rangle$ is well-defined. As it is a continuous linear map, its kernel is a closed hyper-plane and so $(L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}))$ there is $a \in \mathfrak{E}$ such that $f(\eta) = \langle \eta, a \rangle$. Let $N \subseteq \mathbb{N}$ be infinite. Because $\lim \varphi\langle \eta, a - x_i \rangle = \infty$ for all $\eta \in \mathfrak{E}$ it follows by systematic use of the Cauchy-Schwarz inequality that $\{\varphi\langle a - x_i \rangle \mid i \in N\}$ is not bounded above by any $\gamma \in \Gamma$. Therefore $(x_i)_{i \in \mathbb{N}}$ converges to a .

VI. SUFFICIENT CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

VI.1. ASSUMPTIONS. In this chapter $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ is a definite space in the sense of Definition 15. Of the base field k we shall furthermore assume

(cf. Corollaries 22 and 23)

$\Gamma/2\Gamma$ contains a sequence $(\xi_i + 2\Gamma)_{i \in \mathbb{N}}$ such that each

(8) system of representatives $(\xi_i + 2\gamma_i)_{i \in \mathbb{N}}$ that is bounded below tends to ∞ .

(9) k is complete.

Thus, by (8), $\Gamma/2\Gamma$ will be infinite and the topology on k will satisfy the first countability axiom. There are many fields that satisfy (8) and (9): See Remark 30.

The results in the next sections will culminate in Theorem 28 which characterizes certain definite spaces that are orthomodular.

VI.2. COUNTING TYPES. Let $\mathfrak{E}$ be the completion of an $\aleph_0$ -dimensional space $\mathfrak{F}$ which is spanned by an orthogonal basis $\mathcal{B} = (e_i)_{i \in \mathbb{N}}$ that satisfies the type condition (Def. 21). $\mathfrak{F}$ is dense in $\mathfrak{E}$ so $\mathfrak{F}^\perp = (0)$ and hence $\mathcal{B}$ is maximal. By Lemma 16 we have therefore $x = \sum_{i \in \mathbb{N}} \langle x, e_i \rangle \langle e_i \rangle^{-1} e_i$ for all $x \in \mathfrak{E}$.

We now introduce the function v which counts types on $\mathcal{B}$. Let $v: \Gamma/2\Gamma \rightarrow \mathbb{N}: t \mapsto \text{card} \{i \in \mathbb{N} \mid T \circ \phi \langle e_i \rangle = t\}$ (cf. Def. 21). We have

LEMMA 25. If $f_1, \dots, f_m$ are pairwise orthogonal (non zero) vectors in $\mathfrak{E}$ with $T \circ \phi \langle f_i \rangle = t \in \Gamma/2\Gamma$ for all $1 \leq i \leq m$ then $m \leq v(t)$.

Proof. We shall replace the f_i by suitable multiples and assume that $\phi \langle f_i \rangle = \gamma \in \Gamma$ for all $1 \leq i \leq m$. Let $J := \{i \in \mathbb{N} \mid T \circ \phi \langle e_i \rangle = t\}$. We have $f_j = f'_j + f''_j$ where

$$f'_j := \sum_J \langle f_j, e_i \rangle \langle e_i \rangle^{-1} e_i, \quad f''_j := \sum_{\mathbb{N} \setminus J} \langle f_j, e_i \rangle \langle e_i \rangle^{-1} e_i.$$

Since Lemma 14 (ii) generalizes to finite as well as to infinite sums we find $\phi \langle f''_j \rangle = \min_{i \in \mathbb{N} \setminus J} \{\phi \langle \langle f_j, e_i \rangle \langle e_i \rangle^{-1} e_i \rangle\} \neq \phi \langle f_j \rangle$ (because types are different). By Lemma 14 (ii) furthermore $\phi \langle f_j \rangle \leq \phi \langle f'_j \rangle$, $\phi \langle f_j \rangle \leq \phi \langle f''_j \rangle$ and we must have equality in at least one instance. Therefore

$$(10) \quad \phi \langle f_j \rangle = \phi \langle f'_j \rangle = \gamma < \phi \langle f''_j \rangle, \quad 1 \leq j \leq m$$

Now, for $i \neq j$ we find

$$\begin{aligned} 2\phi \langle f'_i, f'_j \rangle &= 2\phi \langle f_i - f''_i, f_j - f''_j \rangle \geq \min \{2\phi \langle f_i, f''_j \rangle, 2\phi \langle f''_i, f_j \rangle, 2\phi \langle f''_i, f''_j \rangle\} \\ &\geq \min \{\phi \langle f_i \rangle + \phi \langle f''_j \rangle, \phi \langle f''_i \rangle + \phi \langle f_j \rangle, \phi \langle f''_i \rangle + \phi \langle f''_j \rangle\} > 2\gamma \end{aligned}$$

so that

$$(11) \quad \varphi\langle f'_i, f'_j \rangle > \gamma, \quad 1 \leq i \neq j \leq m$$

Thus $f'_1, \dots, f'_m$ are an almost orthogonal system in the $v(t)$ -dimensional space $k(e_i)_{i \in J}$. Assume by way of contradiction that the f'_j were linearly dependent, $\sum_1^m \mu_i f'_i = 0$ and not all $\mu_i = 0$. Thus, for each

$$r \in \{1, \dots, m\}, 0 = \sum \mu_i \langle f'_i, f'_r \rangle$$

and so for each r

$$\varphi\langle f'_r \rangle + \varphi(\mu_r) = \varphi\left(-\sum_{j \neq r} \mu_j \langle f'_j, f'_r \rangle\right) \geq \min_{j \neq r} \{\varphi(\mu_j) + \varphi\langle f'_j, f'_r \rangle\}.$$

Therefore, by (10) and (11), $\varphi(\mu_r) > \min_{j \neq r} \{\varphi(\mu_j)\}$ which tells that there is no smallest $\varphi(\mu_r)$ at all, a contradiction. Therefore, $f'_1, \dots, f'_m$ are linearly independent and so $m \leq v(t)$, QED. By Lemma 27 we thus obtain

COROLLARY 26. *The function v that counts types on an orthogonal basis of $\mathfrak{E}$ is the same on all bases.*

VI.3. THE TYPE CONDITION. Let $\mathfrak{E}$ be the completion of a $\aleph_0$ -dimensional space $\mathfrak{F}$ which is spanned by an orthogonal basis $(e_i)_{i \in \mathbb{N}}$ that satisfies the type condition (Def. 21).

LEMMA 27. *Let $\mathcal{B} = (u_i)_{i \in \mathbb{N}}$ be a maximal orthogonal family in $\mathfrak{E}$. Then $\mathcal{B}$ satisfies the type condition and $x = \sum_{i \in \mathbb{N}} \langle x, u_i \rangle \langle u_i \rangle^{-1} u_i$ for all $x \in \mathfrak{E}$. In particular, the span of $\mathcal{B}$ is dense in $\mathfrak{E}$.*

Proof. The assertion on the type condition follows directly from Lemma 25. Let then $x \in \mathfrak{E}$.

$$\begin{aligned} \varphi\langle \langle x, u_i \rangle \langle u_i \rangle^{-1} u_i \rangle &= 2\varphi\langle x, u_i \rangle - \varphi\langle u_i \rangle \geq \varphi\langle x \rangle \\ &+ \varphi\langle u_i \rangle - \varphi\langle u_i \rangle = \varphi\langle x \rangle. \end{aligned}$$

Thus the family of vectors $\langle x, u_i \rangle \langle u_i \rangle^{-1} u_i$ is bounded; in fact, it is a null sequence as $\mathcal{B}$ satisfies the type condition, hence it is summable as $\mathfrak{E}$ is complete. Put $\eta := \sum_{i \in \mathbb{N}} \langle x, u_i \rangle \langle u_i \rangle^{-1} u_i$. We have $\langle u_i, \eta - x \rangle = \langle u_i, \eta \rangle - \langle u_i, x \rangle = 0$, so $x - \eta = 0$ as $\mathcal{B}$ is a maximal orthogonal family. $\square$