# 3. Digression on hyperbolic geometry

proof requires Tits' theorem is due to Gromov: a finitely generated group has polynomial growth if and only if it is almost nilpotent [G].

The analogue of Tits' theorem for division rings does not hold as such [L1], but conjectural statements have been formulated [L2]. Another generalisation of the theorem is proposed as a research problem in remark 1.4.2 of [BL].

## 3. Digression on hyperbolic geometry

Let $n$ be an integer, $n \geqslant 1$. The hyperbolic space $H^{n+1}$ of dimension $n + 1$ is the open unit ball of the euclidean space $\mathbf{R}^{n+1}$. Hyperbolic lines (called lines below) in $H^{n+1}$ are traces on $H^{n+1}$ of circles and euclidean lines in $\mathbf{R}^{n+1}$ which are orthogonal to $\mathbf{S}^n$. Two distinct points $P, Q \in H^{n+1}$ are on a unique line which determines two points $P_\infty, Q_\infty \in \mathbf{S}^n$, say with $P, Q, Q_\infty, P_\infty$ arranged in cyclic order on the euclidean circle defining this line. The (hyperbolic) distance between $P$ and $Q$ is given by a cross-ratio of euclidean distances; more precisely, it is defined to be

$$d(P, Q) = \mathrm{Log}(P, Q, Q_\infty, P_\infty) = \log\left(\frac{|P-Q_\infty|}{|P-P_\infty|} : \frac{|Q-Q_\infty|}{|Q-P_\infty|}\right).$$

The *proper Mœbius group* $GM(n)_0$ is the group of orientation preserving isometries of $\mathbf{R}^{n+1}$ for this distance. Any $g \in GM(n)_0$ extends to a homeomorphism of the closed ball $H^{n+1} \cup S^n$. One may check that $GM(1)_0$ is isomorphic to $PGL(2, \mathbf{R})$ and $GM(2)_0$ to $PGL(2, \mathbf{C})$.

There is an equivalent description with $H^{n+1}$ the half space $\mathbf{R}^n \times \mathbf{R}^*_+$. The set of "points at infinity" is then $\mathbf{R}^n \cup \{\infty\}$ rather than $\mathbf{S}^n$.

For all this, see e.g. [A] or [Si].

An isometry $g \in GM(n)_0$ is said to be

*elliptic* if there is some point in $H^{n+1}$ fixed by $g$,

*parabolic* if there is in $\mathbf{S}^n$ exactly one point fixed by $g$,

*hyperbolic* if there is a line in $H^{n+1}$ invariant by $g$ on which $g$ has no fixed point.

(Following Thurston [Th], we call "hyperbolic" elements which are "loxodromic" in classical litterature, such as in [Gr].)

PROPOSITION. *Elliptic, parabolic and hyperbolic elements define a partition of the proper Mœbius group in three disjoint classes.*

*Proof.* Let us first check that the three classes do not overlap in $GM(n)_0$. If $g$ is hyperbolic, it has two fixed points in $\mathbf{S}^n$ and thus cannot be parabolic; if $g$ was also elliptic, the foot of the perpendicular from the fixed point of $g$ onto the invariant line of $g$ would be fixed by $g$, and this cannot be. If $g$ was at the same time elliptic with fixed point $a \in H^{n+1}$ and parabolic with fixed point $b \in \mathbf{S}^n$, the line from $a$ towards $b$ would have two points at infinity $b$ and $b'$ both fixed by $g$, and this cannot be.

That any $g \in GM(n)_0$ belongs to one of the three classes follows for example from Brouwer's fixed point theorem. (See also 4.9.3 in [Th].) □

Observe that an hyperbolic isometry $g \in GM(n)_0$ has a unique invariant line $\delta$. Suppose indeed that there are two of them, say $\delta$ and $\delta'$. If $\delta \cap \delta' \neq \phi$, the intersection point (which is unique) is fixed by $g$, and this cannot be. If $\delta \cap \delta' = \phi$ and if $\delta$, $\delta'$ have no common point at infinity, there is a unique line perpendicular to both $\delta$ and $\delta'$; but this line intersects $\delta$ in a point fixed by $g$, and this cannot be. Assume finally that $\delta \cap \delta' = \phi$ and that $\delta$ and $\delta'$ have a common point at infinity; choose some number $\rho > 0$ and consider the set $C_\rho$ of points in $H^{n+1}$ at a distance of $\rho$ from $\delta'$; the intersection $C_\rho \cap \delta$ is a point fixed by $g$, and again this cannot be. One may consequently also define an isometry $g \in GM(n)_0$ to be

*elliptic* if $d(a, g(a)) = 0$ for some $a \in H^{n+1}$,

*parabolic* if $\inf d(a, g(a)) = 0$, with the infimum over $a \in H^{n+1}$ not attained,

*hyperbolic* if $\inf d(a, g(a)) > 0$ (and the infimum is then attained exactly on the invariant line of $g$).

We shall need below the following *dynamical description*. An hyperbolic isometry $g \in GM(n)_0$ has on $\mathbf{S}^n$ one attracting point $P_a$ and one repulsing point $P_r$. This means that, for any neighborhood $U$ of $P_a$ in $\mathbf{S}^n$ and for any compact subset $K$ of $S^n - \{P_r\}$, one has $g^k(K) \subset U$ for $k$ large enough. (And similarly with $g^{-1}$ instead of $g$ when exchanging $P_a$ and $P_r$.) Consider now a parabolic isometry $g \in GM(n)_0$ with fixed point $P \in \mathbf{S}^n$. Let $U$ be a neighborhood of $P$ in $\mathbf{S}^n$ and let $K$ be compact in $S^n - \{P\}$; then $g^k(K) \subset U$ for any $k \in \mathbf{Z}$ with $|k|$ large enough. (This is obvious when $g$ is a translation in $\mathbf{R}^n \times \mathbf{R}^*_+$ by some vector in $\mathbf{R}^n$, and any parabolic isometry of $H^{n+1}$ is conjugate to such a translation.)

## 4. Free subgroups of $GL(2, \mathbf{R})$ and of $GL(2, \mathbf{C})$

We show in this section that a subgroup of the proper Mœbius group $G = PGL(2, \mathbf{R})$ which is not almost solvable contains free groups; the same fact for $GL(2, \mathbf{R})$ follows straightforwardly. We discuss also the case of $GL(2, \mathbf{C})$.