Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **29 (1983)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

Introduction

The method of Hadamard and de la Vallée-Poussin arises in the proof that certain classical series, like Riemann's zeta function and Dirichlet's L-functions, do not vanish on the line of absolute convergence. Many interesting equidistribution Theorems are consequences of this result, e.g. the prime number theorem and Dirichlet's Theorem on the infinitude of primes in arithmetic progressions.

Motivated by results of Yoshida [12], Deligne has obtained in his paper *The Weil Conjecture II* ([3], §2) a generalization of the method of Hadamard and de la Vallée-Poussin and has applied it to some very non-classical situations which deal with zeta and L-functions of algebraic varieties over finite fields. Deligne's main result, which is given in Part II and proved in Part III, establishes the non-vanishing on the line of absolute convergence of most of the L-functions which appear naturally in number theory and algebraic geometry; its main merit is its application to L-functions which are not expressible as finite products of Artin L-functions where Brauer induction ordinarily would not suffice.

The present notes, which are an expanded version of the rather concise §2 of [3], have as a purpose to make Deligne's results more accessible to number theorists. We believe that because of its importance the subject deserves a fuller treatment.

In order to reduce the degree of generality in the statement of Deligne's theorem and in his argument, and to give some content to the main result which would be easily understood by number theorists, we start Part I with a series of relatively simple examples taken from elementary algebraic geometry; these are close to the spirit of Artin's thesis [1] as well as that of the beautiful paper of Davenport and Hasse [4]. We hope that the reader will find in Part I some familiar things.

The reader who is only interested in Deligne's Theorem and its proof can consult the last section of Part II and the proof of the main lemma in Part III. In this way he will avoid several excursions that we have taken through the countryside of representation theory. A short sketch of Deligne's application of his result to the proof of the Hard Lefschetz Theorem is given in [6].

We acknowledge several conversations we had with Pierre Deligne about his methods. We also wish to express our deep gratitude to Nick Katz for explaining to us his own ideas on Deligne's results. Without his help and Lecture Notes [5] it would have been almost impossible to write this article. The reader familiar with Katz's Notes (pp. 94-134) will recognize that at times we have followed his

presentation rather closely especially in the proof we give of The Main Lemma in Part III. Most of this article was prepared while the author visited the IHES (1979-80). The present version was presented in three seminars at the University of Illinois in the Spring of 1981.

PART I: EXAMPLES

§1. THE ZETA FUNCTION OF THE PROJECTIVE LINE. Let $\mathbf{F}_q$ be the finite field of q elements and let $A = \mathbf{F}_q[x]$ be the ring of polynomials with coefficients in $\mathbf{F}_q$. The set of closed points on the projective line $\mathbf{P}^1$ can be identified with the set of monic irreducible polynomials in A plus the rational function $\frac{1}{x}$ which corresponds to the point at infinity on $\mathbf{P}^1$. If P is a polynomial in A of degree d, we put

$$NP = q^d .$$

The zeta function of the affine line $\mathbf{A}^1 = \mathbf{P}^1 - \{\infty\}$ is defined, for s a complex number, by

$$Z(s, \mathbf{A}^1) = \sum_a Na^{-s} ,$$

where a runs over all monic polynomials in A including $a = 1$. Since

$$\# \{a \in A \mid a \text{ monic, } \deg(a) = n\} = q^n ,$$

it follows that

$$Z(s, \mathbf{A}^1) = \sum_{n=0}^{\infty} q^{n-ns} = \frac{1}{1 - q^{1-s}} ;$$

hence $Z(s, \mathbf{A}^1)$ is an absolutely convergent series for $R(s) > 1$. Furthermore, since A is a unique factorization domain, we have an Euler product expansion

$$Z(s, \mathbf{A}^1) = \prod_P \frac{1}{1 - NP^{-s}} ,$$

where P runs over all monic irreducible polynomials in A of degree $\geqslant 1$. If we include in this Euler product the factor $(1 - q^{-s})^{-1}$, which corresponds to the rational function $P_\infty = \frac{1}{x}$, we obtain the zeta function of the projective line