10. Deformations of representation homomorphisms and subrepresentations

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So we have the following

(1) $t < i + p(i)$ (by hypothesis)

(2) $n - i \geqslant n + m - t - l$ or equivalently $i \leqslant t + l - m$

(3) $\kappa_m + \ldots + \kappa_{m-l+1} + l \leqslant n + m - t$

(4) $\kappa_1 + \ldots + \kappa_{p(i)} < i \leqslant \kappa_1 + \ldots + \kappa_{p(i)+1}$.

Using (2) and (3) we have that

$$\kappa_m + \ldots + \kappa_{m-l+1} \leqslant n - i = \kappa_1 + \ldots + \kappa_m - i$$

so we have $i \leqslant \kappa_1 + \ldots + \kappa_{m-l}$ which implies $m - l \geqslant p(i) + 1$ thus

$$p(i) + i \leqslant m - l - 1 + i \leqslant (m-l-1) + (t+l-m) = t - 1$$

which contradicts (1). This proves the theorem.

9.7. *Vectorbundles and Schubert cells.* Because every positive vectorbundle over $\mathbf{P}^1(\mathbf{C})$ arises as the bundle $E(\Sigma)$ of some system Σ one has the obvious analogues of theorems 9.5 and 9.6 for positive bundles over $\mathbf{P}^1(\mathbf{C})$. Here the morphism ψ_Σ must, of course, be replaced by the classifying morphism (cf. section 3.2 above) of a positive vector bundle E, and $n + m$ and m are determined respectively as dim $\Gamma(E, \mathbf{P}^1(\mathbf{C}))$ and dim E.

10. Deformations of representation homomorphisms and subrepresentations

10.1 *On proving Inclusion Results for Representations.* Suppose we have given a continuous family of homomorphisms of finite dimensional representations over $\mathbf{C}$ of a finite group G

$$\pi_t : M \to V \tag{10.2}$$

Suppose that $Im\, \pi_t \simeq \rho$ for $t \neq 0$ (and small) and that $Im\, \pi_0 \simeq \rho_0$. Then the representation ρ_0 is a direct summand of the representation ρ. This is seen as follows. Because the category of finite dimensional complex representations of G is semisimple there is a homomorphism of representations $\phi_0 : Im\, \pi_0 \to M$ such that $\pi_0 \circ \phi_0 = id$. Then $\pi_t \circ \phi_0 : Im\, \phi_0 \to Im\, \pi_t$ is still injective for small t (by the continuity of π_t) which gives us ρ_0 as a subrepresentation and hence a direct summand of ρ.

It is almost equally easy to construct a surjective homomorphism $Im\, \pi_t \to Im\, \pi_0$.

10.3. *The Inverse Result.* Inversely if ρ_0 is a subrepresentation of ρ then there is a family of representations (10.3) such that $Im\,\pi_t \simeq \rho$ for $t \neq 0$ and $Im\,\pi_0 \simeq \rho_0$, and if ρ is generated (as a $\mathbf{C}[G]$-module) by one element one can take for M in (10.2) the regular representation. Indeed if ρ_0 is a subrepresentation of ρ then $\rho = \rho_0 \oplus \rho_1$. Let $\pi : M \to \rho = \rho_0 \oplus \rho_1$ be a surjective map of representations. Let π_0, π_1 be the two components of π. Let $s = (s_0, s_1)$ be a section of π. Then $\pi_0 s_0 = id, \pi_1 s_1 = id, \pi_0 s_1 = 0, \pi_1 s_0 = 0$ and it follows that $\pi(t)$ consisting of the components π_0 and $t\pi_1$ is still surjective. Hence $Im\,\pi(t) = \rho$ and $Im\,\pi(0) = \rho_0$.

11. A FAMILY OF REPRESENTATIONS OF S_{n+m} PARAMETRIZED BY $\mathbf{G}_n(\mathbf{C}^{n+m})$

11.1. *Construction of the Family.* Let M be the regular representation of S_{n+m}. That is M has a basis e_σ, $\sigma \in S_{n+m}$ and S_{n+m} acts on M by the formula $\tau(e_\sigma) = e_{\tau\sigma}$, for all $\tau \in S_{n+m}$. Now consider the universal bundle ξ_m over $\mathbf{G}(\mathbf{C}^{n+m})$ and the $n + m$ holomorphic section $\varepsilon_1, \ldots, \varepsilon_{n+m}$ defined by

$$\varepsilon_i(x) = e_i \bmod x \in \mathbf{C}^{n+m}/x ,$$

where e_i is the i-th standard basis vector. Take the $(m+n)$-fold tensor product of ξ_m and define a family of homomorphisms parametrized by $\mathbf{G}_n(\mathbf{C}^{n+m})$ by

$$\pi_x : M \to \xi_m(x)^{\otimes(n+m)},\ e_{\sigma^{-1}} \mapsto \varepsilon_{\sigma(1)}(x) \otimes \ldots \otimes \varepsilon_{\sigma(n)}(x) \tag{11.2}$$

More precisely (11.2) defines a homomorphism of vectorbundles

$$\mathbf{G}_n(\mathbf{C}^{n+m}) \times M \to \xi_m^{\otimes(n+m)} \tag{11.3}$$

The group S_{n+m} acts on $\xi_m(x)^{\otimes(n+m)}$ by permuting the factors and it is a routine exercise to see that π_x is equivariant with respect to this action, i.e. that $\pi_x(\tau v) = \tau\pi_x(v)$ for all $v \in M, \tau \in S_{n+m}$. (Here the product $\tau\sigma \in S_{n+m}$ is interpreted as first the automorphism σ of $1, \ldots, n + m$ and then the automorphism τ.)

Thus $Im\,\pi_x = \pi(x)$ is a representation of S_{n+m} for all x giving us a family of representations parametrized by $\mathbf{G}_n(\mathbf{C}^{n+m})$. Fixing a point $x_0 \in \mathbf{G}_n(\mathbf{C}^{n+m})$ and choosing m independent sections of ξ_m in a neighbourhood U of x_0, this gives us families of homomorphisms of representations