# 8. The Nikodym set

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If we now perform an analogous construction starting from the other side $BC$ of $ABCD$ we obtain a finite number of strips satisfying all the properties indicated (a), (b), (c).

The Besicovitch set is now very easily obtained as follows. We take a square $MNPQ$ of side length 1 and apply to it the above auxiliary construction (**) with $\varepsilon = 1/2$. We obtain a number of strips $\omega_1^1, \omega_2^1, \dots, \omega_{r_1}^1$, covering an area smaller than 1/2 and such that for each segment determined by a point of $MN$ and another of $PQ$ there is a segment of the same length and direction inside $\Omega^1 = \omega_2^1 \cup \omega_2^1 \cup \dots \cup \omega_{r_1}^1$.

Now we consider each of the parallelograms $\omega_j^1$ and apply to it the same construction (**) with $\varepsilon = 1/2^2 r_1$. Collecting all parallelograms corresponding to each $\omega_j^1, j = 1, 2, \dots, r_1$, we obtain a second family of parallelograms $\omega_1^2, \omega_2^2, \dots, \omega_{r_2}^2$. Their union $\Omega^2 = \omega_1^2 \cup \omega_2^2 \cup \dots \cup \omega_{r_2}^2$ has area less than 1/2, is contained in $\Omega^1$ and, again, for each segment joining a point of $MN$ to another of $PQ$ there is another one of the same length and direction inside $\Omega^2$. We proceed with the parallelograms $\omega_j^2$ as we did with the $\omega_j^1$, now with $\varepsilon = 1/2^3 r_2$, and so on. Thus we obtain

$$\Omega^1 \supset \Omega^2 \supset \Omega^3 \supset \dots$$

of areas

$$S(\Omega^1) < 1/2,\ S(\Omega^2) < 1/2^2,\ S(\Omega^3) < 1/2^3, \dots$$

The sets $\Omega^j$ are compact and have the property of containing a parallel translation of each segment with one extremity on $MN$ and the other on $PQ$. The intersection

$$B = \Omega^1 \cap \Omega^2 \cap \dots \cap \Omega^j \cap \dots$$

is of null area and has this same property. We now proceed with the square $MNPQ$ in the same way in the other direction and obtain a compact set of measure zero containing a segment of length one in each direction, i.e. the Besicovitch set.

## 8. The Nikodym set

The Nikodym set can be obtained from the Perron tree in a similar way through the following auxiliary construction, also surprising in itself.

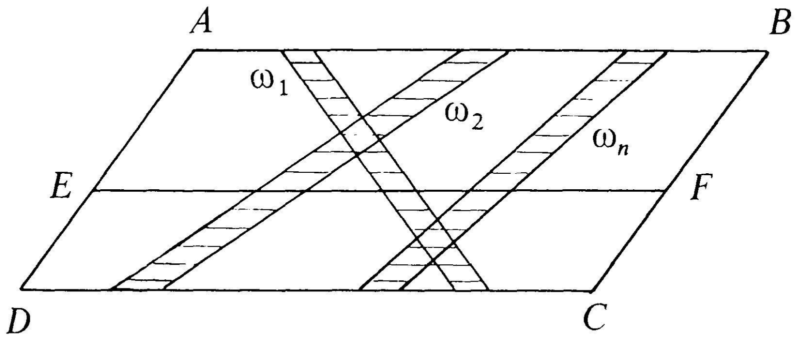


FIGURE 16

*Let $ABCD$ be an arbitrary parallelogram and $CDEF$ another one contained in it as Figure 16 shows. Let $\varepsilon$ be any arbitrary positive number. Then one can construct a finite number of parallelograms $\omega_1, \omega_2, ..., \omega_q$, with a basis on $CD$ and another one on $AB$ such that the figure $\omega_1 \cup \omega_2 \cup ... \cup \omega_q$ covers $CDEF$ while the part of it above $EF$ has area less than $\varepsilon$, that is:*

$$\omega_1 \cup \omega_2 \cup ... \cup \omega_q \supset CDEF$$
$$S((\omega_1 \cup \omega_2 \cup ... \cup \omega_q) \cap (ABCD - CDEF)) < \varepsilon$$

This construction is a little more technical than that of the Besicovitch set and will be omitted. For details we refer to Guzmán (1975).

## 9. MATHEMATICAL FRIVOLITIES? FROM THE PERRON TREE TO THE MEASURE OF THE DENSITY

What started as a puzzle has proved to have many important applications to solve some interesting problems of recent analysis.

Let us assume that we have a mass distributed on the plane and that we wish to measure the density of this distribution at each point. Let us also suppose that the mass is not continuously distributed. One can perhaps say: "Will it not be very artificial to consider a mass that is not continuously distributed?" It is true that the old Scholastic used to affirm that "*natura non facit saltus*" (nature does not proceed by jumps). However, the findings of modern physics permit us to affirm with even stronger motivation "*natura non facit nisi saltus*" (nature proceeds only by jumps). Therefore it is rather natural to consider a discontinuous mass distribution.

For a long time one thought that in order to measure the density one could take *any system of reasonable sets* that contract to the point at which one measures the density, find the mean density over such sets and hope that, when