

# Appendix 1 Relations between polylogarithm and Hurwitz function

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is non-zero according to Dirichlet. Thus we obtain a contribution of  $-1 + \varphi(m)/2$  to the rank coming from the non-trivial even characters.

On the other hand, for the eigenspace corresponding to the trivial character, using formula (10) of §4 we obtain a contribution equal to the number of primes dividing  $m$ . Lemmas 8 and 10 of §5 now complete the proof.  $\square$

## APPENDIX 1

### RELATIONS BETWEEN POLYLOGARITHM AND HURWITZ FUNCTION

For every complex number  $s$ , it follows from Theorem 1 that there exists a linear relation between the even [or the odd] part of the function  $l_s(x)$  and of the function  $\zeta_{1-s}(x)$  or  $\beta_s(x) = -s\zeta_{1-s}(x)$ . This appendix will work out the precise form of these relations. Compare [3], [19], [27].

For integer values of  $s$ , the required relation can be obtained as follows. Recall from formula (9) of §2 that

$$l_0(x) = (-1 + i \cot \pi x)/2$$

hence

$$l_0(x) + l_0(1-x) + \beta_0(x) = 0.$$

Integrating, we see that

$$l_1(x) - l_1(1-x) + 2\pi i \beta_1(x)/1! = 0$$

$$l_2(x) + l_2(1-x) + (2\pi i)^2 \beta_2(x)/2! = 0$$

and so on, for  $0 < x < 1$ . For even values of the subscript, specializing to  $x = 0$  as in §4, this yields Euler's formula

$$2\zeta(2k) + (2\pi i)^{2k} b_{2k}/(2k)! = 0.$$

In particular, it follows that  $\zeta(0) = -\frac{1}{2}$ , and that the numbers  $b_2, -b_4, b_6, -b_8, \dots$  are strictly positive. On the other hand, differentiating the formula for  $l_0(x)$ , we obtain

$$l_{-1}(x) = -\csc^2(\pi x)/4.$$

This is an even function satisfying  $(*_{-1})$ , so it must be some multiple of  $\zeta_2(x) + \zeta_2(1-x)$ . Comparing asymptotic behavior as  $x \rightarrow 0$ , we obtain the classical formula

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x = (2\pi i)^2 l_{-1}(x)/1!.$$

Differentiating, we see that

$$\begin{aligned} -\zeta_3(x) + \zeta_3(1-x) &= (2\pi i)^3 l_{-2}(x)/2! \\ \zeta_4(x) + \zeta_4(1-x) &= (2\pi i)^4 l_{-3}(x)/3! \end{aligned}$$

and so on.

For  $s \neq 0, 1, 2, \dots$  we know from §3 that there is some relation of the form

$$(14) \quad l_s(x) = A_s \zeta_{1-s}(x) + B_s \zeta_{1-s}(1-x)$$

for  $0 < x < 1$ ; where  $A_s$  and  $B_s$  are certain uniquely determined constants. Expressing each of these functions of  $x$  as the sum of an even part and an odd part, we see that

$$(15) \quad \begin{cases} l_s^{\text{even}}(x) = (A_s + B_s) \zeta_{1-s}^{\text{even}}(x) \\ l_s^{\text{odd}}(x) = (A_s - B_s) \zeta_{1-s}^{\text{odd}}(x) . \end{cases}$$

Evidently the functions  $s \mapsto A_s \pm B_s$  are meromorphic, taking finite non-zero values for all  $s \in \mathbf{C} - \mathbf{Z}$ . Differentiating with respect to  $x$ , we see that

$$(16) \quad A_s \pm B_s = s(A_{s+1} \mp B_{s+1})/(2\pi i) .$$

For integral values of  $s$ , using the discussion above, we easily obtain the following table of values, where  $0! = 1$ .

| $s$       | ... | -2                               | -1                              | 0                            | 1                           | 2                               | 3                               | ... |
|-----------|-----|----------------------------------|---------------------------------|------------------------------|-----------------------------|---------------------------------|---------------------------------|-----|
| $1 + B_s$ | ... | 0                                | $\frac{2 \cdot 1!}{(2\pi i)^2}$ | 0                            | $\infty$                    | $\frac{(2\pi i)^2}{2 \cdot 1!}$ | $\infty$                        | ... |
| $1 - B_s$ | ... | $-\frac{2 \cdot 2!}{(2\pi i)^3}$ | 0                               | $-\frac{2 \cdot 0!}{2\pi i}$ | $\frac{2\pi i}{2 \cdot 0!}$ | $\infty$                        | $\frac{(2\pi i)^3}{2 \cdot 2!}$ | ... |

Now suppose that we specialize to  $x = 0$ , by the procedure of §4. Then equation (14) reduces to a form

$$\zeta(s) = (A_s + B_s) \zeta(1-s)$$

of Riemann's functional equation. It follows that

$$(A_s + B_s)(A_{1-s} + B_{1-s}) = 1 ,$$

and hence using (16) that

$$(A_s - B_s)(A_{1-s} - B_{1-s}) = -1 .$$

This discussion gives all of the information about  $A_s \pm B_s$  which we will need. However, it is possible to compute precise values as follows. Let  $\zeta_{1-s}(e^{2\pi i}x)$  be the result of analytic continuation in a loop circling the origin. Then evidently

$$\zeta_{1-s}(e^{2\pi i}x) - \zeta_{1-s}(x) = (e^{2\pi is} - 1)x^{s-1}.$$

Using the integral formula (6), computation shows that

$$l_s(e^{2\pi i}x) - l_s(x) = -(2\pi i)^s x^{s-1} / \Gamma(s).$$

Comparing these two expressions, and noting that  $\zeta_{1-s}(1-x)$  is holomorphic throughout a neighborhood of  $x = 0$ , we can solve for  $A_s$ . The result after some manipulation is

$$A_s = \frac{i(2\pi)^s e^{-\pi is/2}}{2\Gamma(s) \sin(\pi s)}.$$

Now comparing the behavior of  $l_s$  and  $\zeta_{1-s}$  under complex conjugation we see easily that

$$B_s = \overline{A_s} = \frac{-i(2\pi)^s e^{\pi is/2}}{2\Gamma(s) \sin(\pi s)}.$$

In particular, it follows that

$$A_s + B_s = \frac{(2\pi)^s}{2\Gamma(s) \cos(\pi s/2)}, \quad A_s - B_s = \frac{i(2\pi)^s}{2\Gamma(s) \sin(\pi s/2)}.$$

As an application of formula (15), let us prove the corresponding functional equation for a Dirichlet  $L$ -function. Recall from Lemma 14 that for any primitive Dirichlet character  $\chi$  modulo  $m$  the function

$$L(s, \chi) = \sum_1^m \chi(k) \zeta_s(k/m) / m^s$$

satisfies

$$L(s, \bar{\chi}) = \sum_1^m \chi(k) l_s(k/m) / \tau.$$

Here we may just as well use either the even or the odd parts of  $\zeta_s$  and  $l_s$  according as  $\chi(-1)$  is  $+1$  or  $-1$ . Therefore, it follows from (15) that

$$\begin{aligned} L(s, \bar{\chi}) &= (A_s \pm B_s) \sum_1^m \chi(k) \zeta_{1-s}(k/m) / \tau \\ &= m^s (A_s \pm B_s) L(1-s, \chi) / \tau. \end{aligned}$$

Thus we have proved the functional equation

$$(17) \quad L(s, \bar{\chi}) = m^{1-s} (A_s + \chi(-1)B_s) L(1-s, \chi) / \tau(\chi).$$

Here the factor  $m^{1-s}/\tau$  is never zero or infinite, while  $A_s \pm B_s$  is zero or infinite only at certain integer values, as indicated in the table above.

The proof of Lemma 13 can now easily be completed as follows. If  $s \leq 0$  is an integer, then  $L(1-s, \chi) \neq 0$ , so it follows that  $L(s, \bar{\chi})$  equals zero if and only if  $A_s \pm B_s$  is zero, as indicated in the table.  $\square$

## APPENDIX 2

### SOME RELATIVES OF THE GAMMA FUNCTION

This appendix will describe certain functions  $\gamma_1(x)$ ,  $\gamma_2(x)$ , ... which satisfy a modified form of the Kubert identities, with a polynomial correction term. (See (22) below.) They are defined as partial derivatives of the Hurwitz function by the formula

$$(18) \quad \gamma_{1-t}(x) = \partial \zeta_t(x) / \partial t.$$

We will show that  $\gamma_1$  is related to the classical gamma function via Lerch's identity

$$(19) \quad \gamma_1(x) = \log(\Gamma(x)/\sqrt{2\pi}).$$

(Compare [27, p. 60].) As a bonus, we will give a self-contained exposition of the basic properties of the gamma function, based on formulas (18) and (19).

Let us begin with equation (18), which defines  $\gamma_s(x)$  as an analytic function of both variables for all  $s \neq 0$  and all  $x > 0$ . Recall that the Hurwitz function  $\zeta_t(x) = x^{-t} + (x+1)^{-t} + \dots$  (analytically extended in  $t$  for  $t \neq 1$ ) satisfies

$$\zeta_t(x+1) = \zeta_t(x) - x^{-t}.$$

Differentiating with respect to  $t$ , and then substituting  $t = 1 - s$ , we obtain

$$(20) \quad \gamma_s(x+1) = \gamma_s(x) + x^{s-1} \log x.$$

In particular,

$$\gamma_1(x+1) = \gamma_1(x) + \log x.$$

Note that

$$\zeta'_t(x) = -t\zeta_{t+1}(x)$$

hence

$$\zeta''_t(x) = t(t+1)\zeta_{t+2}(x),$$