6.2. The case SU(1, 1)

(b) $\Rightarrow$ (a) : Assume (b). Then A is self-adjoint and positive definite. Define a new inner product $<\cdot,\cdot>$ on $\mathscr{D}(A)$ by $<v, w> := (Av, w)$. Then, for $v, w \in \mathscr{D}(A)$, $g \in G$, we have:

$$\begin{aligned} <\tau(g)v, \tau(g)w> &= \big(A\tau(g)v, \tau(g)w\big) = \big(\tilde{\tau}(g^{-1})A\tau(g)v, w\big) \\ &= \big(A\tau(g^{-1})\tau(g)v, w\big) = (Av, w) = <v, w>, \end{aligned}$$

i.e. $<\tau(g)v, \tau(g)w> = <v, w>$. Thus τ is a unitary representation on $\mathscr{D}(A)$ with respect to the new inner product. (Weak continuity of τ is easily proved.) Let σ be the extension of this representation to a unitary representation in the Hilbert space completion $\mathscr{H}(\sigma)$ of $\mathscr{D}(A)$ with respect to $<\cdot,\cdot>$. Then $\tau \overset{B}{\simeq} \sigma$, where B is the closure of the identity operator on $\mathscr{D}(A)$ (cf. Lemma 4.4). Note that we have also proved the last part of the theorem.

The equivalence of (c) or (d) with (b) follows from Theorem 4.5. □

6.2. The case $SU(1, 1)$

It follows from (2.30) that

$$\overline{c_{\xi,\lambda,n,m}} = (-1)^{m-n}\, c_{\xi,-\bar{\lambda},m,n}\,. \tag{6.4}$$

Combination of (6.3), (2.29) and (6.4) yields

$$\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\bar{\lambda}}\,. \tag{6.5}$$

In §6.1 we showed that a necessary condition for unitarizability of an irreducible subquotient representation τ of $\pi_{\xi,\lambda}$ is the equivalence of τ and $\tilde{\tau}$. In view of (6.5) and Theorem 4.7 this is only possible if $\bar{\lambda} = \pm\lambda$, that is, if λ is real or imaginary. If λ is imaginary then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,\lambda}$, so $\pi_{\xi,\lambda}$ is already unitary. Let us now examine the case that λ is real and nonzero. Then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\lambda}$. If τ is an irreducible subquotient representation of $\pi_{\xi,\lambda}$ then $\tau \overset{A}{\simeq} \tilde{\tau}$ with (cf. (4.10))

$$A\phi_m = c_{\xi,\lambda,m}\,\phi_m,\quad \phi_m \in \mathscr{H}(\tau)\,, \tag{6.6}$$

where $c_{\xi,\lambda,m}$ is given by (4.9). Now a sufficient condition for the unitarizability of τ is that the coefficients $c_{\xi,\lambda,m}$ are all positive or all negative for $\phi_m \in \mathscr{H}(\tau)$. Referring to the classification in Theorem 3.4 we will examine these coefficients. (Because of equivalence, it is not necessary to treat the cases where $\lambda < 0$.)

(a) $$\pi_{0,\lambda}(\lambda>0,\ \lambda\notin\mathbf{Z}+\tfrac{1}{2}).$$

$$c_{0,\lambda,m}=\frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}},\quad m\in\mathbf{Z}.$$

$c_{0,\lambda,m}$ has fixed sign iff $0<\lambda<\frac{1}{2}$.

(b) $$\pi_{\frac{1}{2},\lambda}(\lambda>0,\ \lambda\notin\mathbf{Z}).$$

$$c_{\frac{1}{2},\lambda,m}=\frac{(-\lambda)_{m+\frac{1}{2}}}{(\lambda)_{m+\frac{1}{2}}},\quad m+\tfrac{1}{2}\in\{0,1,2,\ldots\}.$$

No fixed sign.

(c) $$\pi^{+}_{\xi,\lambda}\text{ and }\pi^{-}_{\xi,\lambda}(\lambda+\xi\in\mathbf{Z}+\tfrac{1}{2},\ \lambda>0).$$

$$c_{\xi,\lambda,m}=\frac{(|m|-(\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}},\quad m\in\mathbf{Z}+\xi,\ |m|\geqslant\lambda+\tfrac{1}{2}.$$

Fixed sign.

(d) $$\pi^{0}_{\xi,\lambda}(\lambda+\xi\in\mathbf{Z}+\tfrac{1}{2},\ \lambda>0).$$

$$c_{\xi,\lambda,m}=\frac{(-1)^{m-\xi}}{(\lambda-\frac{1}{2}+m)!(\lambda+\frac{1}{2}-m)!},\quad m\in\left\{-\lambda+\tfrac{1}{2},-\lambda+\tfrac{3}{2},\ldots,\lambda-\tfrac{1}{2}\right\}.$$

No fixed sign except if $\lambda=\frac{1}{2}$, $\xi=0$.

Combining these results with Theorems 3.4, 4.7 and 5.4 and Prop. 4.2 we reobtain BARGMANN's [2] classiffication of all irreducible unitary representations of $SU(1,1)$:

THEOREM 6.2. *Any irreducible unitary representation of $SU(1,1)$ is unitarily equivalent to one and only one of the following representations:*

1) $\pi_{\xi,i\nu}(\xi=0,\frac{1}{2},\ \nu>0)$, $\pi_{0,0}$, $\pi^{+}_{\frac{1}{2},0}$, $\pi^{-}_{\frac{1}{2},0}$ *(unitary principal series).*

2) $\pi_{0,\lambda}(0<\lambda<\frac{1}{2})$ on Cl Span$\{\ldots,\phi_{-1},\phi_0,\phi_1,\ldots\}$

with respect to the inner product

$$<\phi_m\,\phi_n>:=\frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}}\,\delta_{m,n}\ \textit{(complementary series).}$$

3) $$\pi^{+}_{\xi,\lambda}\ \textit{and}\ \pi^{-}_{\xi,\lambda}\left(\xi=0\text{ or }\tfrac{1}{2},\ \lambda=\xi+\tfrac{1}{2},\xi+\frac{3}{2},\ldots\right)$$

on

$$Cl\ \mathrm{Span}\{\phi_{\lambda+\frac{1}{2}}, \phi_{\lambda+3/2}, \ldots\}$$

and

$$Cl\ \mathrm{Span}\{\ldots, \phi_{-\lambda-3/2}, \phi_{-\lambda-\frac{1}{2}}\},$$

respectively, with respect to the inner product

$$<\phi_m, \phi_n> : = \frac{(|m| - (\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}} \delta_{m,n} \text{ (discrete series).}$$

4) $\pi^0_{0,\frac{1}{2}}$ *(identity representation).*

6.3. Notes

6.3.1. Following BARGMANN [2], most authors prove Theorem 6.2 by infinitesimal methods. VILENKIN [43, Ch. VI] uses the method of the present paper. TAKAHASHI [39, §6] decides about unitarizability by considering whether $\pi_{\xi,\lambda,n,n}$ is a positive definite function on G.

6.3.2. A method related to this section was used in FLENSTED-JENSEN & KOORNWINDER [15] in order to find all irreducible unitary spherical representations of non-compact semisimple Lie groups G of rank one. They examined the nonnegativity of the coefficients in the addition formula for the spherical functions on G. See also [27, §6.4].

6.3.3. A generalization of Theorem 6.1 can be formulated for not necessarily abelian K and, partly, for K-finite τ, cf. [27, Theorems 6.4, 6.5].