2. The canonical matrix elements of the principal series

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

$SU(n, 1)$. For convenience, in order to avoid matrix manipulations, we restrict ourselves here to the case that the compact subgroup K is abelian.

The results of this paper may be generalized rather easily to the universal covering group of $SL(2, \mathbf{R})$. The extension to $SL(2, \mathbf{C})$ was done by KOSTERS [28], see also NAIMARK [34, ch. 3, §9]. Hopefully, an extension to $SO_0(n, 1)$ and $SU(n, 1)$ is feasible.

The reader of this paper is supposed to already have a modest knowledge about certain elements of semisimple Lie theory, like principal series and spherical functions. Suitable references will be given. Some of this preliminary material can also be found in the earlier version [27]. Modern accounts of the infinitesimal approach to $SL(2, \mathbf{R})$ can be found, for instance, in SCHMID [36, §2] or VAN DIJK [9]. TAKAHASHI [42] also presented a global approach to $SL(2, \mathbf{R})$, partly based on an earlier version of the present paper, partly (the global proof of Theorem 5.4) independently.

Finally, I would like to thank G. van Dijk and M. Flensted-Jensen for useful comments.

2. THE CANONICAL MATRIX ELEMENTS OF THE PRINCIPAL SERIES

2.1. PRELIMINARIES

Let G be a locally compact group satisfying the second axiom of countability (lcsc. group). A *Hilbert representation* of G is a strongly continuous but not necessarily unitary representation τ of G on some Hilbert space $\mathscr{H}(\tau)$ (which is always assumed to be separable). Let K be a compact subgroup of G. A Hilbert representation τ of G is called *K-unitary* if the restriction $\tau|_K$ of τ to K is a unitary representation of K. A Hilbert representation τ of G is called *K-finite* respectively *K-multiplicity free* if τ is K-unitary and each $\delta \in \hat{K}$ has finite multiplicity respectively multiplicity 1 or 0 in $\tau|_K$. If τ is K-multiplicity free then the *K-content* $\mathscr{M}(\tau)$ of τ is the set of all $\delta \in \hat{K}$ which have multiplicity 1 in $\tau|_K$.

Let K be a compact abelian subgroup of G and let τ be a K-multiplicity free representation of G. Choose an orthogonal basis $\{\phi_\delta \mid \delta \in \mathscr{M}(\tau)\}$ of $\mathscr{H}(\tau)$ such that

$$\tau(k)\phi_\delta = \delta(k)\phi_\delta, \qquad \delta \in \mathscr{M}(\tau), k \in K.$$

We call $\{\phi_\delta\}$ a *K-basis* for $\mathscr{H}(\tau)$ and the functions $\tau_{\gamma\delta}(\gamma, \delta \in \mathscr{M}(\tau))$, defined by

$$\tau_{\gamma,\delta}(g) := (\tau(g)\phi_\delta, \phi_\gamma), \quad g \in G, \tag{2.1}$$

the *canonical matrix elements* of τ (with respect to K).

2.2. The principal series

Let G be a connected noncompact real semisimple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition. For $g \in G$ write $g = u(g)\exp(H(g))n(g)$, where $u(g) \in K$, $H(g) \in \mathfrak{a}$ (the Lie algebra of A) and $n(g) \in N$. Let $\rho \in \mathfrak{a}^*$ be half the sum of the positive roots. Let M be the centralizer of A in K. For $\xi \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ the *principal series* representation $\pi_{\xi,\lambda}$ of G is obtained by inducing the (not necessarily unitary) finite-dimensional irreducible representation $man \to e^{\lambda(\log a)}\xi(m)$ of the subgroup MAN. In the so-called compact picture we have the following realization of $\pi_{\xi,\lambda}$ (cf. Wallach [45, §8.3]):

$$(2.2)\qquad \begin{gathered}\left(\pi_{\xi,\lambda}(g)f\right)(k) = e^{-(\rho+\lambda)(H(g^{-1}k))}\, f\left(u(g^{-1}k)\right),\\ f \in L^2_\xi\left(K, \mathscr{H}(\xi)\right), \qquad k \in K, g \in G.\end{gathered}$$

Here the Hilbert space $L^2_\xi(K, \mathscr{H}(\xi))$ consists of all $\mathscr{H}(\xi)$-valued L^2-functions f on K such that $f(km) = \xi(m^{-1})f(k)$, $k \in K$, $m \in M$. The representation $\pi_{\xi,\lambda}$ is a K-unitary Hilbert representation. It is unitary if $\lambda \in i\mathfrak{a}^*$. By Frobenius reciprocity, $\pi_{\xi,\lambda}$ is K-finite and $\pi_{\xi,\lambda}$ is K-multiplicity free if each $\delta \in \hat{K}$ is M-multiplicity free.

Let us now specialize the above results to $G = SL(2, \mathbf{R})$. It is convenient to work with the group $G = SU(1, 1)$, isomorphic to $SL(2, \mathbf{R})$:

$$(2.3)\qquad G := \left\{ g_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix};\quad \alpha, \beta \in \mathbf{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Let

$$(2.4)\qquad K := \left\{ u_\theta = \begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{pmatrix};\quad 0 \leqslant \theta < 4\pi \right\},$$

$$(2.5)\qquad A := \left\{ a_t = \begin{pmatrix} ch\frac{1}{2}t & sh\frac{1}{2}t \\ sh\frac{1}{2}t & ch\frac{1}{2}t \end{pmatrix} = \exp t \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}; t \in \mathbf{R} \right\},$$

$$(2.6)\qquad N := \left\{ n_z = \begin{pmatrix} 1+\frac{1}{2}iz & -\frac{1}{2}iz \\ \frac{1}{2}iz & 1-\frac{1}{2}iz \end{pmatrix}; z \in \mathbf{R} \right\}.$$

Then $G = KAN$ is an Iwasawa decomposition for $G = SU(1, 1)$, $\rho(\log a_t) = \frac{1}{2}t$ and $M = \{u_0, u_{2\pi}\}$. $\hat{M}$ consists of the two one-dimensional representations

$$(2.7) \qquad u_\theta \to e^{i\xi\theta}, \quad u_\theta \in M, \xi = 0 \text{ or } \tfrac{1}{2}.$$

Let $L^2_\xi(K)$ consist of all $f \in L^2(K)$ such that $f(u_{\psi+2\pi}) = f(u_\psi)$ or $-f(u_\psi)$ according to whether $\xi = 0$ or $\frac{1}{2}$, respectively.

Now, by using explicit expressions for the factors in the Iwasawa decomposition of $g_{\alpha,\beta}^{-1} u_\psi$ (cf. TAKAHASHI [39, §1]) we can write (2.2) in the case G $SU(1, 1)$ as follows:

$$(2.8) \qquad \big(\pi_{\xi,\lambda}(g_{\alpha,\beta})f\big)(u_\psi) := |\bar{\alpha}e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}|^{-2\lambda-1} f(u_{\psi'}),$$

$$\psi' := 2\arg(\bar{\alpha}e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}), \quad g_{\alpha,\beta} \in G, u_\psi \in K, f \in L^2_\xi(K),$$

$$\xi = 0 \text{ or } \tfrac{1}{2}, \lambda \in \mathbf{C}.$$

On putting $g_{\alpha,\beta} := u_\theta \in K$ we get

$$(2.9) \qquad \big(\pi_{\xi,\lambda}(u_\theta)f\big)(u_\psi) = f(u_{\psi-\theta}), \quad f \in L^2_\xi(K), u_\theta, u_\psi \in K,$$

which again shows that $\pi_{\xi,\lambda}$ is K-unitary. $\hat{K}$ consists of the representations

$$(2.10) \qquad \delta_n(u_\theta) := e^{in\theta}, \quad u_\theta \in K,$$

where n runs through the set $\frac{1}{2}\mathbf{Z}$, i.e., $2n \in \mathbf{Z}$. An orthogonal basis for $L^2_\xi(K)$ is given by the functions

$$(2.11) \qquad \phi_n(u_\psi) := e^{-in\psi}, \quad u_\psi \in K,$$

where n runs through the set $\mathbf{Z} + \xi := \{m+\xi \mid m \in \mathbf{Z}\}$. Then

$$(2.12) \qquad \pi_{\xi,\lambda}(u_\theta)\phi_n = \delta_n(u_\theta)\phi_n, \quad u_\theta \in K, n \in \mathbf{Z} + \xi.$$

Thus $\pi_{\xi,\lambda}$ is K-multiplicity free,

$$(2.13) \qquad \mathscr{M}(\pi_{\xi,\lambda}) = \{\delta_n \in \hat{K} \mid n \in \mathbf{Z} + \xi\},$$

the ϕ_n's form a K-basis for $L^2_\xi(K)$ and the canonical matrix elements of $\pi_{\xi,\lambda}$ are

$$(2.14) \qquad \pi_{\xi,\lambda,m,n}(g) = (\pi_{\xi,\lambda}(g)\phi_n, \phi_m), \quad g \in G, m, n \in \mathbf{Z} + \xi.$$

Because of the Cartan decomposition $G = KAK$, $\pi_{\xi,\lambda,m,n}$ is completely determined by its restriction to A. It follows from (2.8) and (2.11) that

$$\big(\pi_{\xi,\lambda}(a_t)\phi_n\big)(u_\psi) = |ch\tfrac{1}{2}t\, e^{\frac{1}{2}i\psi} - sh\tfrac{1}{2}t\, e^{-\frac{1}{2}i\psi}|^{-2\lambda+2n-1}$$

$$\cdot (ch\tfrac{1}{2}t\, e^{\frac{1}{2}i\psi} - sh\tfrac{1}{2}t\, e^{-\frac{1}{2}i\psi})^{-2n}.$$

Hence

$$\pi_{\xi,\lambda,m,n}(a_t) = (ch\tfrac{1}{2}t)^{-2\lambda-1} \tag{2.15}$$

$$\cdot \frac{1}{4\pi}\int_0^{4\pi} (1-th\tfrac{1}{2}t\, e^{i\psi})^{-\lambda+n-1/2}(1-th\tfrac{1}{2}t\, e^{-i\psi})^{-\lambda-n-1/2}e^{i(m-n)\psi}d\psi\,,$$

$$t \in \mathbf{R},\ m,n \in \mathbf{Z} + \xi\,.$$

The following symmatry is evident from (2.15):

$$\pi_{\xi,\lambda,-m,-n}(a_t) = \pi_{\xi,\lambda,m,n}(a_t)\,. \tag{2.16}$$

2.3. Calculation of the canonical matrix elements

Let us calculate the integral (2.15). In view of (2.16) we can suppose $m \geqslant n$. The binomial expansion

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad |z| < 1, a \in \mathbf{C}\,, \tag{2.17}$$

where

$$(a)_k := a(a+1)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}\,, \tag{2.18}$$

can be substituted for the first two factors in the integrand of (2.15). Now interchange the order of summation and integration and perform the integration in each term. Then we obtain $(m \geqslant n)$

$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{n-m-2\lambda-1} \tag{2.19}$$

$$\cdot\, {}_2F_1(\lambda+m+\tfrac{1}{2}, \lambda-n+\tfrac{1}{2}; m-n+1; (th\tfrac{1}{2}t)^2)\,,$$

where the ${}_2F_1$ denotes a *hypergeometric series*, defined by

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k, \quad |z| < 1, a,b,c \in \mathbf{C}\,, \tag{2.20}$$

cf. [10, Vol. I, Ch. 2].

The expression (2.20) is clearly symmetric in a and b. As a function of z, the ${}_2F_1$ has an analytic continuation to a one-valued function on $\mathbf{C}\backslash[1, \infty)$. Application of the transformation formulas

$$(2.21) \qquad {}_2F_1(a, b; c; z) = (1-z)^{-b}{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

(cf. [10, Vol. I, §2.1 (22)]) to (2.19) yields ($m \geqslant n$):

$$(2.22) \qquad \pi_{\xi, \lambda, m, n}(a_t)$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{-m-n}{}_2F_1(\lambda-n+\tfrac{1}{2}, -\lambda-n+\tfrac{1}{2}; m-n+1; -(sh\tfrac{1}{2}t)^2)$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{m+n}{}_2F_1(\lambda+m+\tfrac{1}{2}, -\lambda+m+\tfrac{1}{2}; m-n+1; -(sh\tfrac{1}{2}t)^2).$$

It is more elegant to express the hypergeometric functions in (2.22) in terms of *Jacobi functions* $\phi_\mu^{(\alpha,\beta)}$ ($\mu, \alpha, \beta \in \mathbf{C}$, $\alpha \notin \{-1, -2, \ldots\}$), which are defined on $\mathbf{R}$ by

$$(2.23) \qquad \phi_\mu^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\mu), \tfrac{1}{2}(\alpha+\beta+1-i\mu); \alpha+1; -(sht)^2)$$

(cf. KOORNWINDER [36, §2]). Clearly,

$$(2.24) \qquad \phi_\mu^{(\alpha,\beta)}(0) = 1,$$

$$(2.25) \qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_\mu^{(\alpha,\beta)}(-t),$$

$$(2.26) \qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_{-\mu}^{(\alpha,\beta)}(t).$$

The function $\phi_\mu^{(\alpha,\beta)}$ satisfies the differential equation

$$(2.27) \qquad (\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}\left(\Delta_{\alpha,\beta}(t)\frac{du(t)}{dt}\right) = -(\mu^2+(\alpha+\beta+1)^2)u(t),$$

where

$$\Delta_{\alpha,\beta}(t) := (sht)^{2\alpha+1}(cht)^{2\beta+1},$$

and $u := \phi_\mu^{\alpha,\beta}$ is the unique solution of (2.27) which is regular at $t = 0$ and satisfies $u(0) = 1$. For fixed $\alpha > -1$, $\beta \in \mathbf{R}$, Jacobi functions $\phi_\mu^{(\alpha,\beta)}$ form a continuous orthogonal system with respect to the measure $\Delta_{\alpha,\beta}(t)dt$, $t > 0$.

Substitution of (2.23) and (2.22) yields $(m \geq n)$:

(2.28)
$$\pi_{\xi,\lambda,m,n}(a_t)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)\,!}\,(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{-m-n}\phi_{2i\lambda}^{(m-n,\,-m-n)}(\tfrac{1}{2}t)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)\,!}\,(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(m-n,\,m+n)}(\tfrac{1}{2}t)\,.$$

Application of (2.16) gives a similar result in the case $m < n$. Finally we conclude:

THEOREM 2.1. *The canonical matrix elements* $\pi_{\xi,\lambda,m,n}(a_t)$ $(\lambda \in \mathbf{C};\ \xi = 0$ or $\frac{1}{2};\ m, n \in \mathbf{Z} + \xi;\ t \in \mathbf{R})$ *of* $SU(1,1)$ *can be expressed in terms of Jacobi functions by*

(2.29)
$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{c_{\xi,\lambda,m,n}}{(|m-n|)\,!}\,(sh\tfrac{1}{2}t)^{|m-n|}\,(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(|m-n|,\,m+n)}(\tfrac{1}{2}t)\,,$$

where

(2.30)
$$c_{\xi,\lambda,m,n} := \begin{cases}(\lambda+n+\frac{1}{2})_{m-n} & \text{if } m \geq n\,,\\ (\lambda-n+\frac{1}{2})_{n-m} & \text{if } n \geq m\,.\end{cases}$$

In view of (2.24), formulas (2.29) and (2.30) describe the asymptotics of $\pi_{\xi,\lambda,m,n}$ near $t = 0$.

2.4. NOTES

2.4.1. The principal series of representations was first written down for $SL(2, \mathbf{R})$ by BARGMANN [2], for $SL(2, \mathbf{C})$ by GELFAND & NAIMARK [18], and for a general noncompact semisimple Lie group by HARISH-CHANDRA [21, §12].

2.4.2. BARGMANN [2, §10] already obtained explicit expressions in terms of hypergeometric functions for the canonical matrix elements of the irreducible unitary representations of $SL(2, \mathbf{R})$. He solved the differential equation satisfied by these matrix elements, which is obtained from the Casimir operator. VILENKIN [43, Ch. VI, §3] gives a derivation of these expressions which is similar to our derivation in §2.4, starting from the integral representation (2.15).

2.4.3. It follows from the present paper that the spherical functions for $SL(2, \mathbf{R})$ can be expressed as Jacobi functions of order $(\alpha, \beta) = (0, 0)$. More generally, the spherical functions on any noncompact real semisimple Lie group of rank 1 (i.e., $\dim(A) = 1$) can be written as Jacobi functions of certain order (cf. HARISH-CHANDRA [23, §13]). This motivated FLENSTED-JENSEN [14] to study harmonic analysis for Jacobi function expansions of quite general order (α, β), $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. This research was continued in several papers by Flensted-Jensen and the author.

3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

3.1. SUBQUOTIENT REPRESENTATIONS

We start with the definition and some general properties and next derive an irreducibility criterium (Theorem 3.2) and a decomposition theorem 3.3.

Let G be a lcsc. group and let τ be a Hilbert representation of G. Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}(\tau)$ and let P_0 be the orthogonal projection from $\mathscr{H}(\tau)$ onto $\mathscr{H}_0$. Define

$$\tau_0(g)v := P_0\tau(g)v\,, \quad g \in G, v \in \mathscr{H}_0\,. \tag{3.1}$$

Then $\tau(g) \in \mathscr{L}(\mathscr{H}_0)$ for each $g \in G$, $\tau_0(e) = id.$, and $g \to \tau_0(g)v : G \to \mathscr{H}_0$ is continuous for each $v \in \mathscr{H}_0$. If also

$$\tau_0(g_1 g_2) = \tau_0(g_1)\tau_0(g_2)\,, \quad g_1, g_2 \in G\,, \tag{3.2}$$

then τ_0 is a Hilbert representation of G on $\mathscr{H}_0$ and it is called a *subquotient representation* of τ. Formula (3.2) is clearly valid if $\mathscr{H}_0$ is an *invariant subspace* of $\mathscr{H}(\tau)$, i.e., if $\tau(g)v \in \mathscr{H}_0$ for all $g \in G$, $v \in \mathscr{H}_0$. In that case, τ_0 is called a *subrepresentation* of τ.

LEMMA 3.1. *Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}(\tau)$, let $\mathscr{H}_2$ be the closed G-invariant subspace of $\mathscr{H}(\tau)$ which is generated by $\mathscr{H}_0$ and let $\mathscr{H}_1 := \mathscr{H}_2 \cap \mathscr{H}_0^{\perp}$. Then τ_0 is a subquotient representation if and only if $\mathscr{H}_1$ is G-invariant.*