

# §1. Introduction

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# ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. III

by Karl K. NORTON

## §1. INTRODUCTION

Let  $P$  be the set of all (positive rational) prime numbers, and let  $E$  be an arbitrary nonempty subset of  $P$ . Throughout this paper, let  $p$  denote a general member of  $P$ , and for non-negative integers  $a$ , write  $p^a \parallel n$  if  $p^a \mid n$  and  $p^{a+1} \nmid n$ . For each positive integer  $n$ , define

$$\omega(n; E) = \sum_{p \mid n, p \in E} 1, \quad \Omega(n; E) = \sum_{p^a \parallel n, p \in E} a.$$

We usually write  $\omega(n; P) = \omega(n)$ ,  $\Omega(n; P) = \Omega(n)$ . In this paper, we shall estimate the functions

$$S(x, y; E, \omega) = \text{card} \{n \leq x : \omega(n; E) > y\}, \quad (1.1)$$

$$S(x, y; E, \Omega) = \text{card} \{n \leq x : \Omega(n; E) > y\}$$

when  $y$  is appreciably larger than the normal order of  $\omega(n; E)$  and  $\Omega(n; E)$ ;  $y$  may even be as large as the maximum order of  $\omega(n; E)$  or  $\Omega(n; E)$ , respectively. (Here and throughout,  $\text{card } B$  means the number of members of the set  $B$ , and if  $Q(n)$  is a statement about the integer  $n$ , we often write  $\{n \leq x : Q(n)\}$  instead of  $\{n : 1 \leq n \leq x \text{ and } Q(n)\}$ .)

Define

$$E(x) = \sum_{p \leq x, p \in E} p^{-1} \quad (x \text{ real}). \quad (1.2)$$

In [13], it was observed that if  $E(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$ , then both the average order and the normal order of  $\omega(n; E)$  are equal to  $E(x)$ , and the same statement holds for  $\Omega(n; E)$ . In [13], we obtained sharp inequalities for the functions (1.1) when  $0 < y < 2E(x)$ , roughly. In [14], we gave asymptotic formulas for the same functions when  $E(x) \rightarrow +\infty$  and  $y = E(x) + o(E(x))$  as  $x \rightarrow +\infty$ . It is well-known, however, that

$$E(x) \leq \log \log x + O(1) \quad \text{for } x \geq 2,$$

whereas if  $x$  is large,  $\omega(n; E)$  and  $\Omega(n; E)$  may be much larger than  $\log \log x$  for some values of  $n \leq x$ . For example, the method of [6, pp. 262-263, 359] shows that

$$\limsup_{n \rightarrow +\infty} \frac{\omega(n) \log \log n}{\log n} = 1, \quad (1.3)$$

and a more precise version of (1.3) was obtained in [12, pp. 96-100]. (See also the remarks at the beginning of §3 below.) Before stating estimates for the functions (1.1) when  $y$  is large, it seems worthwhile to generalize results like (1.3) to  $\omega(n; E)$ . First define

$$\pi(x; E) = \sum_{p \leq x, p \in E} 1 \quad (x \text{ real}), \quad (1.4)$$

and write

$$\begin{aligned} \log_2 x &= \log \log x, & \log_r x &= \log(\log_{r-1} x) \\ \text{for } r &= 3, 4, \dots \end{aligned} \quad (1.5)$$

**THEOREM 1.6.** *Suppose that there exists a real number  $\gamma(E) > 0$  such that*

$$\begin{aligned} \pi(x; E) &= \gamma(E) (x/\log x) \{1 + O_E(1/\log x)\} \\ \text{for all } x &\geq 2. \end{aligned} \quad (1.7)$$

*Then for each  $n \geq 3$ , we have*

$$\begin{aligned} \omega(n; E) &\leq \frac{\log n}{\log_2 n} + \frac{\{1 + \log \gamma(E)\} \log n}{(\log_2 n)^2} \\ &\quad + O_E\left(\frac{\log n}{(\log_2 n)^3}\right), \end{aligned} \quad (1.8)$$

*with equality for infinitely many  $n$ .*

Here and throughout, the notation  $O_{\delta, \varepsilon, \dots}$  implies a constant depending at most on  $\delta, \varepsilon, \dots$ , while  $O$  without subscripts implies an *absolute* constant. Likewise, for  $i = 1, 2, \dots$ , we shall write  $c_i(\delta, \varepsilon, \dots)$  for a positive number depending at most on  $\delta, \varepsilon, \dots$ , while  $c_i$  will mean a positive absolute constant.

It is interesting to observe that a much weaker hypothesis than (1.7) still implies that the maximum order of  $\omega(n; E)$  is approximately  $(\log n)(\log_2 n)^{-1}$ . See the remarks after the proof of Theorem 1.6 in §3.

After (1.3) and Theorem 1.6, it is natural to ask how often  $\omega(n; E)$  and  $\Omega(n; E)$  assume values appreciably larger than their normal order  $E(n)$ . It appears that rather little was known about this problem until very recently. The earliest contribution was by Hardy and Ramanujan [5] (reprinted in [15,

pp. 262-275]), whose estimate for  $\text{card } \{n \leq x : \omega(n) = m\}$  leads easily to a good upper bound for  $S(x, y; P, \omega)$  (essentially the same as the bound given in Theorem 1.14 below). However, they did not state explicitly a result of the latter type. For arbitrary  $E$ , much weaker upper bounds for  $S(x, y; E, \omega)$  and  $S(x, y; E, \Omega)$  can be derived from a general theorem of Turán [19] on the distribution of values of additive functions. (See also Turán [18] or Hardy and Wright [6, pp. 356-358] for the case  $E = P$ , and see [13, §§1, 3] and [14, pp. 18-19] for remarks on all of this early work.) For the particular functions  $\omega(n; E)$  and  $\Omega(n; E)$ , Turán's bounds were improved considerably in the author's paper [13; (5.16), (5.15), (1.11)], where it was observed that for any set  $E$ ,

$$S(x, \alpha E(x); E, \omega) \leq x \exp \{(\alpha - 1 - \alpha \log \alpha) E(x)\} \quad (1.9)$$

for real  $x \geq 1$ ,  $\alpha \geq 1$ , where  $E(x)$  is defined by (1.2). A similar (slightly less precise) result was stated for  $\Omega(n; E)$  when  $1 \leq \alpha < p_1$ , where  $p_1$  is the smallest member of  $E$ . No lower bound was obtained in either case for  $\alpha \geq 2$ , so that the precision of (1.9) for large  $\alpha$  was not clear. In a later paper [2], Erdős and Nicolas obtained a rather good estimate in the special case  $E = P$ . They showed that for any fixed  $\alpha$  with  $0 < \alpha < 1$ ,

$$\text{card } \{n \leq x : \omega(n) > \alpha (\log x) (\log_2 x)^{-1}\} = x^{1-\alpha+o(1)} \quad (1.10)$$

as  $x \rightarrow +\infty$ . (In fact, they obtained a somewhat more precise result resembling Theorem 4.13 below.) However, they did not get an analogous result for  $\Omega(n)$ , nor did they generalize to  $\omega(n; E)$  or  $\Omega(n; E)$ . Furthermore, their method did not give good upper estimates for  $S(x, y; P, \omega)$  when  $y$  is appreciably smaller than  $(\log x) (\log_2 x)^{-1}$ . We propose to remedy all of these drawbacks to some extent. First, we obtain the following lower bound by a refinement of the Erdős-Nicolas method:

**THEOREM 1.11.** *Suppose that there exists a real number  $\gamma(E) > 0$  such that (1.7) holds. Let  $\varepsilon > 0$ , and suppose that  $x \geq c_1(E, \varepsilon)$  and*

$$c_2(E) \leq y \leq (\log x) (\log_2 x)^{-1} + \{1 + \log \gamma(E) - \varepsilon\} (\log x) (\log_2 x)^{-2}. \quad (1.12)$$

*Then*

$$S(x, y; E, \omega) \geq x \exp \{-y (\log y + \log_2 y - \log \gamma(E) - 1) + O_E(y (\log_2 y) / \log y)\}. \quad (1.13)$$

(1.8) shows that only a very small weakening of the hypothesis (1.12) would be of any interest. In Theorem 3.20, we assume much less than (1.7) and derive a result similar to Theorem 1.11 (but somewhat weaker).

Concerning upper bounds for  $S(x, y; E, \omega)$ , we have obtained only a modest improvement of (1.9); see Theorem 4.8 and Corollary 4.12. It should be emphasized that (1.9) and Theorem 4.8 hold for an arbitrary set  $E$  (without the assumption (1.7)). Using the same methods, we deduce

**THEOREM 1.14.** *Suppose that there exists a real number  $\gamma(E) > 0$  such that (1.7) holds. If  $x \geq 3$  and  $y \geq \gamma(E) \log_2 x$ , then*

$$S(x, y; E, \omega) \leq x \exp \{ -y (\log y - \log_3 x - \log \gamma(E) - 1) - \gamma(E) \log_2 x + O_E(y/\log_2 x) \}. \quad (1.15)$$

Although there is a considerable gap between (1.13) and (1.15), the results are more general and somewhat sharper than those of Erdős and Nicolas [2]. In particular, we get a generalization of (1.10) (see Theorem 4.13). Theorems 1.11 and 1.14 also yield immediately the following result which could not be obtained by the Erdős-Nicolas method:

**COROLLARY 1.16.** *Suppose that there exists a real number  $\gamma(E) > 0$  such that (1.7) holds. If  $0 < \alpha < 1$  and  $x \geq c_3(E, \alpha)$ , then*

$$S(x, (\log x)^\alpha; E, \omega) = x \exp \{ -\alpha (\log x)^\alpha \log_2 x + O((\log x)^\alpha \log_3 x) \}.$$

It should be mentioned that when  $E = P$  (the set of all primes) and  $y/\log_2 x$  is bounded and not too close to 1, Theorems 1.11 and 1.14 can be replaced by a striking asymptotic formula which was recently obtained by H. Delange (for the proof, see [2]):

**THEOREM 1.17 (Delange).** *Let  $x, \alpha, r_1, r_2$  be real with  $x \geq 3$ ,  $1 < r_1 \leq \alpha \leq r_2$ . Then*

$$S(x, \alpha \log_2 x; P, \omega) = \frac{F(\alpha) \alpha^{1/2 + \alpha \log_2 x - [\alpha \log_2 x]}}{(2\pi)^{1/2} (\alpha - 1)} \cdot \frac{x}{(\log x)^{1 - \alpha + \alpha \log \alpha} (\log_2 x)^{1/2}} \left\{ 1 + O_{r_1, r_2} \left( \frac{1}{\log_2 x} \right) \right\},$$

where  $[z]$  means the largest integer  $\leq z$  and

$$F(\alpha) = \frac{1}{\Gamma(\alpha+1)} \prod_p \left(1 + \frac{\alpha}{p-1}\right) \left(1 - \frac{1}{p}\right)^\alpha.$$

Delange obtained a similar result for card  $\{n \leq x: \omega(n) \leq \alpha \log_2 x\}$  when  $x \geq 3$ ,  $(\log_2 x)^{-1} \leq \alpha \leq r_3 < 1$  (see [2]). In this connection, it is interesting to note the estimate

$$F(\alpha) = \exp \{-\alpha \log \alpha - \alpha \log_2 \alpha + (1-\gamma)\alpha + O(\alpha/\log \alpha)\}$$

for real  $\alpha \geq 2$ , where  $\gamma$  is Euler's constant. (Some effort is required to show this, and we omit the proof.)

For values of  $\alpha$  near 1, Kubilius [8, Theorem 9.2] proved a result on the distribution of  $\omega(n)$  which is similar to Theorem 1.17. His theorem was later extended by himself [9] and Laurinćikas [10] to somewhat more general additive functions, and it was generalized to  $\omega(n; E)$  and  $\Omega(n; E)$  by Norton [14]. The estimates for  $S(x, y; E, \omega)$  derived in the present paper are not as precise as Theorem 1.17 or the earlier work cited, but they are more general with respect to  $E$  (except for [14]), and they hold for much larger values of  $y$ .

We now consider the function  $\Omega(n; E)$ . Here we assume that  $E$  is any nonempty set of primes; in particular, nothing like (1.7) is assumed. For completeness, we begin by stating the following easy result:

**THEOREM 1.18.** *Let  $p_1$  be the smallest member of  $E$ . Then*

$$\Omega(n; E) \leq (\log n) (\log p_1)^{-1} \quad \text{for all } n \geq 1, \quad (1.19)$$

with equality if and only if  $n = p_1^a$  for some integer  $a \geq 0$ .

This follows from

$$n \geq \prod_{p^a \parallel n, p \in E} p^a \geq \prod_{p^a \parallel n, p \in E} p_1^a = p_1^{\Omega(n; E)}.$$

We now proceed to estimate  $S(x, y; E, \Omega)$  (defined by (1.1)). For  $y \geq 2E(x)$ , rather little previous work has been done on this problem, and all of it was restricted to the special case  $E = P$  (the set of all primes). Selberg [17, p. 87] stated without detailed proof the following asymptotic formula:

$$\text{card } \{n \leq x: \Omega(n) = m\} \sim A 2^{-m} x \log x$$

for integers  $m$  satisfying  $(2+\varepsilon) \log_2 x \leq m \leq B \log_2 x$ . (Here  $\varepsilon > 0$  is arbitrarily small, while  $A$  and  $B$  are positive absolute constants; it is not clear

from [17] how large  $B$  could be.) Selberg also gave an asymptotic formula for  $\text{card } \{n \leq x : \omega(n) = m\}$  when  $m/\log_2 x$  is bounded. His work was recently extended to considerably larger values of  $m$  (roughly  $m < (\log x)^{3/5}$ ) by Kolesnik and Straus [7], whose theorems are quite complicated. These results, together with the formula

$$S(x, y; P, \Omega) = \sum_{y < m \leq Y} \text{card } \{n \leq x : \Omega(n) = m\} + S(x, Y; P, \Omega)$$

and different tools for estimating  $S(x, Y; P, \Omega)$  from above, would yield some information about  $S(x, y; P, \Omega)$ . However, it appears that neither [17] nor [7] would thus lead to an estimate for  $S(x, y; P, \Omega)$  which is both simple and reasonably precise when  $y/\log_2 x$  is unbounded. To the best of our knowledge, the only previous result of the latter type is due to Erdős and Sárközy [3], who recently proved that

$$S(x, y; P, \Omega) \leq c_4 y^4 2^{-y} x \log x \quad \text{for } x \geq 3, y \geq 1. \quad (1.20)$$

We shall generalize their work to  $S(x, y; E, \Omega)$  and get a sharper upper bound. Although the result could be phrased in terms of the function  $E(x)$  (defined by (1.2)), it is more convenient to state it in terms of a real number  $v$  which in practice is taken to be an approximation to  $E(x)$ . (For example, if  $E = P$ , we could take  $v = \log_2 x$ .)

**THEOREM 1.21.** *Let  $x, v, y$  be real with  $x \geq 1, v \geq 1$ , and  $y \geq 0$ . Let  $p_1$  be the smallest member of  $E$ , and define*

$$\Lambda = \Lambda(x, v; E) = \max \{2, |E(x) - v|\}. \quad (1.22)$$

*Then*

$$S(x, y; E, \Omega) \leq c_5 (p_1) p_1^{-y} x v^{1/2} e^{(p_1 - 1)v + p_1 \Lambda}. \quad (1.23)$$

We remark that (1.23) is our best upper bound when  $y > p_1 v - v^{1/2}$ , but it can be improved for smaller values of  $y$  (see Lemma 5.3).

Concerning the problem of estimating  $S(x, y; E, \Omega)$  from below, we shall state only the following simple result:

**THEOREM 1.24.** *Let  $p_1$  be the smallest member of  $E$ . If  $x \geq p_1$  and  $0 \leq y \leq (\log x)(\log p_1)^{-1} - 1$ , then*

$$S(x, y; E, \Omega) \geq (1/2) p_1^{-y-1} x.$$

To prove this, let  $k = [y] + 1$  (so  $k$  is the smallest integer greater than  $y$ ), and observe that the multiples  $n$  of  $p_1^k$  have the property that  $\Omega(n; E) \geq k > y$ .

There are just  $[xp_1^{-k}]$  of these  $n \leq x$ , and since  $[z] \geq z/2$  for  $z \geq 1$ , we get the result.

It is clear that Theorem 1.24 is essentially best possible in certain extreme cases (for example, if  $E = \{p_1\}$ , or if  $x = p_1^a$  and  $y = a - 1$ ).

When  $E = P$  (the set of all primes), we can take  $v = \log_2 x$ . Then  $\Lambda = O(1)$ , and we have the following corollary of Theorems 1.21 and 1.24:

**COROLLARY 1.25.** *If  $x \geq e^e$  and  $0 \leq y \leq (\log x)(\log 2)^{-1} - 1$ , then*

$$2^{-y-2} x \leq S(x, y; P, \Omega) \leq c_6 2^{-y} x (\log x) (\log_2 x)^{1/2}.$$

Corollary 1.25 should be compared with the Erdős-Sárközy result (1.20) and with the asymptotic formula of Selberg mentioned after Theorem 1.18. When  $y < 2 \log_2 x$  (roughly), more precise estimates for  $S(x, y; P, \Omega)$  can be obtained from [13] and [14].

In a later paper, we shall show that if  $p_1$  is the smallest member of  $E$  and  $\varepsilon > 0$  is fixed, then the precise order of magnitude of  $S(x, y; E, \Omega)$  is

$$p_1^{-y} x \exp \{(p_1 - 1) E(x)\}$$

when  $E(x)$  is sufficiently large and

$$p_1 E(x) \leq y \leq (1 - \varepsilon) (\log x) (\log p_1)^{-1}.$$

This theorem is much more difficult to prove than Theorem 1.21. Its proof depends on Theorem 1.21 and on an extension of Halász's work [4] concerning the local distribution of  $\Omega(n; E)$ . Theorem 1.21 remains our best upper bound when  $y$  is close to  $(\log x)(\log p_1)^{-1}$  (cf. Theorem 1.18), and it seems to be the most we can achieve by a fairly simple method.

## §2. NOTATION

The symbols  $a, m, n$  always represent integers with  $a \geq 0, m \geq 0, n > 0$ . The letter  $p$  always denotes a prime, while  $v, w, x, y, z, \alpha, \beta, \delta, \varepsilon, \sigma$  are real numbers.  $[x]$  means the largest integer  $\leq x$ . The notation  $\log x$  is defined by (1.5), and the notations  $O, O_{\delta, \varepsilon, \dots}, c_i, c_i(\delta, \varepsilon, \dots)$  are explained after Theorem 1.6. If a condition such as " $x \geq c_i(\delta, \varepsilon, \dots)$ " is used as a hypothesis, it is to be understood that  $c_i(\delta, \varepsilon, \dots)$  is sufficiently large. We shall occasionally use the notations  $\ll, \gg$  to imply constants which are *absolute*. (Thus  $A = O(B)$  is equivalent to  $A \ll B$ .)