# 3. Representations of non-compact semisimple groups

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In section 2 we interpreted the equation

$$\text{(3.1)} \qquad \mathrm{Hom}_G\left(V, H^{0,j}_{\bar\partial}(G/P, E_W)\right) = \mathrm{Hom}_{p \cap \bar p}\left(W^*, H^j(n, V^*)\right)$$

in (2.16) as a precise extension of Frobenius reciprocity to higher cohomology; also see (2.8), (2.12), and (2.14). There we used $\bar\partial$ cohomology and assumed that our space $G/P = K/K \cap P$ was compact. In this section we shall see yet another extension of Frobenius reciprocity where $K$ is replaced by a real non-compact semisimple group and $\bar\partial$ cohomology is replaced by "square integrable" $\bar\partial$ cohomology. In this non-compact context the *discrete series* defined in the introduction will play the analogous role of the dual objects $\hat K$ of equivalence classes of irreducible unitary representations of $K$. The analogue of the generalized Borel-Weil theorem (Theorem 2.25) for example will be formulated for the discrete series. This is Schmid's solution of the Kostant-Langlands conjecture.

In this section $G$ will now denote a real non-compact connected semisimple Lie group with finite center and $K$ will denote a maximal compact subgroup of $G$. If $\pi$ is any unitary representation of $G$ on a Hilbert space $H$ and $f \in L_1(G)$ we let

$$\text{(3.2)} \qquad \pi(f) = \int_G f(x)\, \pi(x)\, dx$$

so that $\pi(f)$ is a bounded operator on $H$ satisfying $\pi(f * g) = \pi(f)\,\pi(g)$ for $f$, $g \in L_1(G)$. $*$ denotes convolution and $dx$ denotes Haar measure on the unimodular group $G$. The following fundamental theorem of Harish-Chandra is valid ([20]).

Theorem 3.3. *If $\pi$ is an irreducible unitary representation of $G$ and $f \in C_c^\infty(G)$ is a compactly supported smooth function on $G$ then the operator $\pi(f)$ is of trace class. Moreover the equation $\Theta_\pi(f) = \mathrm{trace}\, \pi(f)$, $f \in C_c^\infty(G)$, defines a distribution $\Theta_\pi$ on $G$ in the sense of L. Schwartz. $\Theta_\pi$ depends only on the unitary equivalence class $[\pi]$ of $\pi$. For two such classes $[\pi_1], [\pi_2]$ we have $\Theta_{[\pi_1]} = \Theta_{[\pi_2]}$ if and only if $[\pi_1] = [\pi_2]$.*

The distribution $\Theta_\pi$ is called the (global) *character* of $\pi$ (or of $[\pi]$). The fact that $\pi(f)$ is of trace class is a consequence of the following fundamental deep fact: There is an integer $N \geqslant 1$ such that for any irreducible unitary representations $\pi$, $\sigma$ of $G$ and $K$ we have

$$\text{(3.4)} \qquad \text{the multiplicity of } \sigma \text{ in } \pi \Big|_K \leqslant N \text{ dimension of } \sigma .$$

$\Theta_\pi$ is invariant under all inner automorphisms of $G$ and if $\mathscr{Z}$ is the algebra of bi-invariant differential operators [1]) on $G$ then the space of distributions $\{Z\,\Theta_\pi \mid Z \in \mathscr{Z}\}$ has dimension one. That is $\Theta_\pi$ is $\mathscr{Z}$-finite and is in fact an *eigendistribution* of $\mathscr{Z}$. By Harish-Chandra's profound regularity theorem for invariant $\mathscr{Z}$-finite distributions [24] one may conclude that $\Theta_\pi$ is a locally integrable function on $G$ which is actually analytic on the regular points $G'$ of $G$. $G'$ is an open dense set in $G$ such that the complement $G - G'$ has Haar measure zero.

We now recall Harish-Chandra's character formula for the discrete series. We assume that $G$ admits a Cartan subgroup $H$ such that $H \subset K$. From the introductory remarks we recall that this assumption guarantees precisely that $G$ has a discrete series. In order to avoid certain technical difficulties we shall assume moreover for simplicity that $G$ is linear and that its complexification $G^{\mathbf{C}}$ is simply connected. Let $g$, $k$, $h$ denote the complexifications of the Lie algebras $g_0$, $k_0$, $h_0$ of $G$, $K$, $H$ respectively. As in section 2 we let $\Delta$ denote the set of non-zero roots of $(g, h)$ and we let $2\delta = \sum_{\alpha \in \Delta^+} \alpha$ for a choice of positive system of roots $\Delta^+ \subset \Delta$. Again we say that $\Lambda \in h^*$ is *integral* if $2\dfrac{(\Lambda, \alpha)}{(\alpha, \alpha)}$ is an integer for each $\alpha$ in $\Delta$. Thus $\exp X \to e^{\Lambda(x)}$ is a well defined character of $H$, $x \in h_0$.

THEOREM 3.5 (Harish-Chandra, 1964). *Let $\Lambda \in h^*$ be integral and suppose that $(\Lambda + \delta, \alpha) \neq 0$ for every $\alpha$ in $\Delta$* [2]*). Then there exists a discrete series representation $\pi_\Lambda$ of $G$ such that*

$$\text{(3.6)} \qquad \Theta_{\pi_\Lambda}(\exp X) = \frac{(-1)^m \operatorname{sgn} \prod_{\alpha \in \Delta^+} (\Lambda + \delta, \alpha) \sum_{\sigma \in W_K} \det \sigma \, e^{[\sigma(\Lambda + \delta)](x)}}{\prod_{\alpha \in \Delta^+} \left(e^{\alpha(x)/2} - e^{-\alpha(x)/2}\right)}$$

[1]) We may identify $\mathscr{Z}$ with the center of the universal enveloping algebra of the complexified Lie algebra of $G$.

[2]) That is $\Lambda + \delta$ is regular.

for $x \in h_0$, where $2m = \dim G/K$ and $W_K$ is the Weyl group of $(K, H)$. Moreover every discrete series representation is equivalent to some $\pi_\Lambda$ where $\Lambda \in h^*$ is integral and $\Lambda + \delta$ is regular and $\pi_{\Lambda_1}$ is equivalent to $\pi_{\Lambda_2}$ if and only if $\sigma(\Lambda_1 + \delta) = \Lambda_2 + \delta$ for some $\sigma$ in $W_K$ (see [26]).

The factor $(-1)^m \operatorname{sgn} \prod_{\alpha \in \Delta^+} (\Lambda + \delta, \alpha)$ in (3.6) can be expressed in an alternate form: Let $\Delta_n$, $\Delta_k$ denote the set of non-compact and compact roots respectively. Thus by definition $\alpha \in \Delta_n$ (or $\Delta_k$) if $g_\alpha \subset p_0^{\mathbf{C}}$ (or if $g_\alpha \subset k_0^{\mathbf{C}} = k$) where

(3.7)
$$g_0 = k_0 + p_0$$

is a Cartan decomposition of $g_0$. Let $\Delta_n^+ = \Delta^+ \cap \Delta_n$ and $\Delta_k^+ = \Delta^+ \cap \Delta_k$. Then it is easy to check that

(3.8)
$$(-1)^m \operatorname{sgn} \prod_{\alpha \in \Delta^+} (\Lambda + \delta, \alpha) = (-1)^{q_\Lambda}$$

where

(3.9)
$$q_\Lambda \overset{\text{def.}}{=} |\{\alpha \in \Delta_n^+ \mid (\Lambda + \delta, \alpha) > 0\}| + |\{\alpha \in \Delta_k^+ \mid (\Lambda + \delta, \alpha) < 0\}|$$

and $|S|$ is the cardinality of a set $S$. One notes the similarity in appearance of Harish-Chandra's character formula (3.6) and Weyl's character formula of Theorem 2.22.

For an integral $\Lambda \in h^*$ such that $\Lambda + \delta$ is regular we continue to denote the corresponding discrete series representation of Theorem 3.5 by $\pi_\Lambda$. Let $\mathscr{L}_\Lambda \to G/H$ denote the $C^\infty$ line bundle over $G/H$ induced by the character (which we have seen is well-defined) $\exp x \to e^{\Lambda(x)}$ of $H$, $x \in h_0$. Let $P$ denote the Borel subgroup of $G^{\mathbf{C}}$ corresponding to the subalgebra $p = h + \sum_{\alpha \in \Delta^+} g_{-\alpha}$ of $g$. Then $G \cap P = H$ so that by general principles, see [40], [78], $G/H$ is an open $G$ orbit in $G^{\mathbf{C}}/P$ and thus $G/H$ has a $G$ invariant complex structure such that, at the origin, $n = \sum_{\alpha \in \Delta^+} g_{-\alpha}$ is the space of anti-holomorphic tangent vectors; and moreover $\mathscr{L}_\Lambda$ also has a holomorphic structure [1]). $G/H$ may therefore be considered as a non-compact analogue of the space $K/K \cap P$ of section 2 with $\mathscr{L}_\Lambda$ playing the analogous role of $E_\Lambda$. Thus given the Borel-Weil theorem (Theorem 2.25) one naturally inquires whether the representation $\pi_\Lambda$ occurs on a $\bar{\partial}$ cohomology space of differential forms with coefficients in $\mathscr{L}_\Lambda$. This question was posed (more precisely) first by B. Kostant and R. Langlands in 1965. Since $G/H$ is non-compact we should consider $L_2$-cohomology. Namely we proceed as

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[1]) The $G$ invariant complex structures on $G/H$ correspond to the choices of positive root systems $\Delta^+$.

follows. Let $\Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)$ denote the space of complex smooth compactly supported forms of type $(0, j)$ on $G/H$ with values in $\mathscr{L}_\Lambda$; cf. remarks preceding (2.10). With respect to natural $G$ invariant hermitian metrics on the fibres of $\mathscr{L}_\Lambda$ and on the tangent bundle of $G/H$ $\Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)$ has the structure of a complex inner product space and the Cauchy-Riemann operator

$$\bar{\partial} : \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda) \to \Lambda_c^{0,j+1}(G/H, \mathscr{L}_\Lambda)$$

has a formal adjoint

$$\bar{\partial}^* : \Lambda_c^{0,j+1}(G/H, \mathscr{L}_\Lambda) \to \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)\,.$$

As usual let

$$\Box = \bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\,\bar{\partial} : \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda) \\ \to \Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)$$

denote the corresponding complex Laplace-Beltrami operator. Let $L_2^{0,j}(G/H, \mathscr{L}_\Lambda)$ denote the Hilbert space completion of $\Lambda_c^{0,j}(G/H, \mathscr{L}_\Lambda)$ and set

$$(3.10) \qquad H_{\bar{\partial},2}^{0,j}(G/H, \mathscr{L}_\Lambda) \\ = \{\phi \in L_2^{0,j}(G/H, \mathscr{L}_\Lambda) \mid \Box\,\phi = 0 \text{ in the distribution sense}\}\,.$$

$H_{\bar{\partial},2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ is called the $L_2$-cohomology (or harmonic) space attached to the Hermitian line bundle $\mathscr{L}_\Lambda$. The reader should consult [63] for a more general definition of $L_2$-cohomology spaces [1]). We take this opportunity to point out that in [63] line 9 of page 96 should be corrected to read $\vartheta = (\bar{\partial})^*$ rather than $\vartheta = (\bar{\partial}_0)^*$.

One knows that $H_{\bar{\partial},2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ is a closed subspace of

$$L_2^{0,j}(G/H, \mathscr{L}_\Lambda)$$

and hence it is a Hilbert space. Moreover since the above hermitian metrics were chosen to be $G$ invariant, $H_{\bar{\partial},2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ carries a natural unitary representation of $G$-namely that induced by left translation on forms; cf. (2.13). We denote this representation by $\pi_\Lambda^{0,j}$. The analogies with section 2 presist since in the compact case classical Hodge theory represents the $\bar{\partial}$ cohomology in (2.11) as solutions of a complex Laplacian . □

Now Kostant and Langlands conjectured, in analogy with Theorem 2.25, that the spaces $H_{\bar{\partial},2}^{0,j}(G/H, \mathscr{L}_\Lambda)$ should vanish for each $j$ if $\Lambda + \delta$ is not regular and if $\Lambda + \delta$ is regular (as we have assumed) then one should have vanishing for

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[1]) The definition (3.10) suffices for our purposes since we are dealing with a homogeneous space $G/H$ where the metric is therefore automatically complete.

all but one $j$, say $j = j_0$. Moreover the representations $\pi_\Lambda^{0, j_0}$ and $\pi_\Lambda$ should coincide; see [51], [54]. Based on a vanishing theorem of P. Griffiths [40] some results of Harish-Chandra [22] and a formal application of the Wood's Hole fixed point formula [54], Langlands predicted moreover that the value of $j_0$ should be the number $q_\Lambda$ given in (3.9). Important progress towards the verification of the Kostant-Langlands conjecture was made by W. Schmid in his 1967 thesis [77] and by Schmid [78], M. S. Narasimhan, and K. Okamoto in 1969 [60]. In 1973 W. Casselman and M. Osborne proved a version of Kostant's theorem on $n$ cohomology (see Theorem 2.21) in the case where the coefficient module (for $g$) is *infinite dimensional* but has an infinitesimal character [17]. Schmid used the Casselman-Osborne result decisively in [82] and thereby settled the conjecture:

THEOREM 3.11 (W. Schmid, 1975). *Let* $\Lambda \in h^*$ *be an integral element. If* $(\Lambda + \delta, \alpha) = 0$ *for some root* $\alpha$ *then the space*

$$H^{0, j}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda)$$

*in (3.10) vanishes for each* $j$. *Suppose that* $\Lambda + \delta$ *is regular. Let* $\pi_\Lambda$ *be the corresponding discrete series representation whose character is given by (3.6) in Theorem 3.5. Let* $q_\Lambda$ *be the integer defined in (3.9). Then*

$$H^{0, j}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda) = 0$$

*for* $j \neq q_\Lambda$ *and the natural unitary representation* $\pi_\Lambda^{0, q_\Lambda}$ *of* $G$ *on*

$$H^{0, q_\Lambda}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda)$$

*is irreducible and equivalent to* $\pi_\Lambda$.

*Remarks.* In the preceding we have assumed for simplicity that $G$ was linear. It is now known that this assumption can be dropped in the statement of Theorem 3.11. We may also drop the integrality of $\Lambda$ in the statement of Theorem 3.11 and assume, more generally, only that $\Lambda$ is real-valued on roots and extends to a character of $H$. It is then still true that $\mathscr{L}_\Lambda$ carries a holomorphic structure; see remark 3 in [78].

The point we wish to stress now is that just as Theorem 2.25 is derivable from the Frobenius reciprocity relation (3.1), or (equivalently) from (2.18), Theorem 3.11 is like-wise derivable from a non-compact analogue of (2.18). In [78] (lemma 6) Schmid obtains a direct integral decomposition of the $L^2$-harmonic spaces $H^{0, j}_{\bar{\partial}, 2}(G/H, \mathscr{L}_\Lambda)$ using the Plancherel decomposition of $L^2(G)$:

$$H^{0,j}_{\bar{\partial},2}(G/H, \mathscr{L}_\Lambda) = \int_{\hat{G}} V_\pi^* \otimes \mathscr{H}^j(\pi)_{e^{-\Lambda}}\, d\pi \tag{3.12}$$

where $\mathscr{H}^j(\pi)$ is a certain *formal harmonic space* attached to the irreducible unitary representation $\pi \in \hat{G}$ of $G$ and $\mathscr{H}^j(\pi)_{e^{-\Lambda}}$ is the subspace of $\mathscr{H}^j(\pi)$ transforming under the action of $H$ according to the character $e^{-\Lambda}$. The tensor product in (3.12) is a tensor product of Hilbert spaces; $V_\pi^*$ is the contragradient representation space of $(\pi, V_\pi) \in \hat{G}$ (where $\pi$ acts on the Hilbert space $V_\pi$). Given $(\pi, V_\pi) \in \hat{G}$, $\mathscr{H}^j(\pi)$ is defined as follows. First let $V_\pi^\infty$ denote the space of $K$ finite vectors in $V_\pi$ (these are just the vectors in $V_\pi$ whose $K$ translates span a finite-dimensional subspace). In the usual way, via differentiation, $V_\pi^\infty$ is a $g$ module (the induced action is skew-hermitian) and by restriction (as in section 2) $V_\pi^\infty$ is an $n$ module where we take

$$n = \sum_{\alpha \in \Delta^+} g_{-\alpha} \tag{3.13}$$

$n$ has a natural $\mathrm{Ad}_H$ invariant inner product induced by the Killing form of $g$; see equation (2.4) of [78]. Thus we may consider the formal adjoint $\delta^*$ of the Lie algebra coboundary operator

$$\delta\colon V_\pi^\infty \otimes \Lambda n^* \to V_\pi^\infty \otimes \Lambda n^*$$

corresponding to the $n$ module $V_\pi^\infty$. One has

THEOREM 3.14 (Lemma 3 of [78]). *$\delta + \delta^*$ has a unique self-adjoint extension $\overline{\delta + \delta^*}$. Also $\overline{\delta + \delta^*}$ is the only closed extension of $\delta + \delta^*$.*

$V_\pi^\infty \otimes \Lambda^j n^*$ is dense in $V_\pi \otimes \Lambda^j n^*$ for each $j$ (since $V_\pi^\infty$ is dense in $V_\pi$). By definition $\mathscr{H}^j(\pi)$ is the kernel of $\overline{\delta + \delta^*}$ considered as a closed densely defined operator in $V_\pi \otimes \Lambda^j n^*$. The $H$ action on the cochains $V_\pi^\infty \otimes \Lambda^j n^*$ commutes with $\delta$ and also with $\delta^*$ (since the $n$ inner product is $\mathrm{Ad}_H$ invariant) which means that $\mathscr{H}^j(\pi)$ (and the Lie algebra cohomology $H^j(n, V_\pi^\infty)$) inherits an $H$ module structure. The subspace $\mathscr{H}^j(\pi)_{e^{-\Lambda}}$ in (3.12) of vectors in $\mathscr{H}^j(\pi)$ transforming according to the character $e^{-\Lambda}$ under this $H$ action is therefore well-defined.

In [82] (Theorem 3.1) Schmid proves the $H$ module isomorphism

$$\mathscr{H}^j(\pi) \simeq H^j(n, V_\pi^\infty) \tag{3.15}$$

From the Casselman-Osborn result [17] (which as we have pointed out is a version of Kostant's Theorem 2.21 for infinite dimensional $g$ modules with an infinitesimal character $-V_\pi^\infty$ being such an example),

$$H^j(n, V_\pi^\infty)_{e^\mu} = 0$$

unless $V_\pi^\infty$ has a specific infinitesimal character (this means that on $V_\pi^\infty$ the center of the universal enveloping algebra $Ug$ of $g$ must act by a specific scalar). In Harish-Chandra's notation [20] this character is $\chi_{-\mu-\delta}$, where again $2\delta = \sum_{\alpha \in \Delta^+} \alpha$; here $\mu \in h^*$ is integral or, more generally, $\mu$ is real-valued on roots and defines a character of $H$ (see remarks following Theorem 3.11). On the other hand, from the harmonic analysis of $G$ it is known that only finitely many irreducible unitary equivalence classes can have a fixed infinitesimal character and that moreover if $F \subset \hat{G}$ is a finite set which is disjoint from the classes of discrete series, then the Plancherel measure must vanish on $F$. Thus from these observations one concludes from (3.15) that only discrete series modules $(\pi, V_\pi)$ can occur in the direct integral decomposition given in (3.12) and since the $(\pi, V_\pi)$ occur discretely we obtain (cf. Corollary 3.23 of [82]) the following refinement of (3.12).

THEOREM 3.16 (Frobenius-Schmid reciprocity, 1975). *As G modules*

$$H^{0,j}_{\bar{\partial},2}(G/H, \mathscr{L}_\Lambda) = \sum_{\substack{(\pi, V) = \\ \text{discrete class}}} V_\pi^* \otimes H^j(n, V_\pi^\infty)_{e^{-\Lambda}}$$

This is the non-compact analogue of (2.18) (where the contragradient $W^*$ of the inducing module $W$ there is replaced by the contragradient $e^{-\Lambda}$ of the inducing character $e^\Lambda$). Theorem 3.16 preceeds and implies (with the knowledge of $n$ cohomology, as in the compact case) Theorem 3.11.

## 4. REMARKS ON THE NILPOTENT CASE: POLARIZATIONS AND HARMONIC INDUCTION

The Frobenius reciprocity in higher cohomology discussed in the two preceding sections extends to a non-semisimple Lie group context as well. Moreover consequent analogues of the Kostant-Langlands conjecture have been proved. Most recently (within the past few months) remarkable and complete results along these lines have been obtained (independently) for simply connected nilpotent Lie groups by J. Rosenberg [74] and R. Penney [69]. Their results are preceded by results of H. Moscovici and A. Verona [59]; also see [15], [58], [62], [67], [68], [75]. In this regard one of the central notions to consider is that of a *polarization*. It is defined as follows. Let $g$ be a real Lie algebra, let $\Lambda \in g^*$