§2. The case of generalized flag manifolds

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$$\begin{tikzcd}
X_P \arrow[rrr, "\tilde{f}_1^{-1}"] \arrow[dr, "\mathrm{can}"'] & & & X_P \arrow[dl, "\mathrm{can}"] \\
& X_0 \arrow[r, "f_1^{-1}"] & X_0 & \\
X_{\bar{P}} \arrow[ur, "\gamma"] \arrow[rrr, "\tilde{f}_2"'] & & & X_{\bar{P}} \arrow[ul, "\mathrm{can}"']
\end{tikzcd}$$

which shows that $Y \simeq X$.

§2. The case of generalized flag manifolds

The following result is an easy consequence of [F].

LEMMA 2.1. Let M be a generalized flag manifold. Then the following holds.

a) If $g(\lambda) \in Gr(M_0)$ is a grading map with $\lambda \in \mathbf{Z}_Q^*$ for some (not necessarily finite) set of primes Q, then $g(\lambda)$ lifts to a homotopy equivalence $\tilde{g}(\lambda): M_Q \to M_Q$.

b) Let P be an arbitrary set of primes with complement $\bar{P}$. Then every

$$f \in \langle Gr(M_0), N(H)/H \rangle$$

may be written in the form $f = f_1 \circ f_2$ with

$$f_1 \in \operatorname{im}\big(E(M_P) \to E(M_0)\big)$$

and

$$f_2 \in \operatorname{im}\big(E(M_{\bar{P}}) \to E(M_0)\big).$$

Proof. Let $\lambda = k/l$ with k and l relatively prime integers. Then $g(k)$ and $g(l)$ lift to equivalences

$$\tilde{g}(k), \tilde{g}(l): M_Q \to M_Q$$

since necessarily $k, l \in \mathbf{Z}_Q^*$ (compare [F]). Thus $\tilde{g}(k)\,\tilde{g}(l)^{-1}$ is a lift of $g(\lambda)$. For b) we note that $f = g(\rho) \circ \sigma$ for some $\rho \in \mathbf{Q}^*$ and

$$\sigma \in N(H)/H .$$

If we write $\rho = \rho_1 \cdot \rho_2$ with $\rho_1 \in \mathbf{Z}_P^*$ and $\rho_2 \in \mathbf{Z}_{\bar{P}}^*$, then

$$f = g(\rho_1) \cdot \big(g(\rho_2)\,\sigma\big)$$

and we may choose

$$f_1 = g(\rho_1),\; f_2 = g(\rho_2)\,\sigma .$$

Since σ lifts even to $E(M)$, we infer by using a) that f_1 and f_2 lift as desired.

A final step towards proving the Theorem formulated in the introduction consists in the following.

LEMMA 2.2. Let M be a generalized flag manifold for which Conjecture C holds. Then for every finite set of primes P,

$$P\text{-Seq}\,\big(E(M_0)\big) = \{[1, 1, ..., 1]\} .$$

Proof. Let $\{[\mu_1, ..., \mu_n]\} \in P$-Seq $\big(E(M_0)\big)$, where $P = \{p_1, ..., p_n\}$ and

$$\mu_i \in \operatorname{im}\big(E(M_{p_i}) \to E(M_0)\big)$$

for all i. Then $\mu_i = g(\lambda_i) \circ \sigma_i$ with $\lambda_i \in \mathbf{Q}^*$ and

$$\sigma_i \in N(H)/H \subset E(M_0) .$$

Define $\lambda \in \mathbf{Q}^*$ by $\lambda = \Pi p_i^{m_i}$, where $m_i \in \mathbf{Z}$ is such that $p_i^{m_i}\lambda_i \in \mathbf{Z}_{\bar{p}_i}^*$. Then $g(\lambda)\,\mu_i = g(\lambda\lambda_i)\,\sigma_i$ with $\lambda\lambda_i \in \mathbf{Z}_{\bar{p}_i}^*$. By Lemma 2.1 a) we know that $g(\lambda\lambda_i)$ lifts to M_{p_i}, and since σ_i lifts even to M we conclude that

$$h(p_i) = g(\lambda\lambda_i)\,\sigma_i \in \operatorname{im}\big(E(M_{p_i}) \to E(M_0)\big)$$

for all i. The equation

$$g(\lambda)\,\mu_i = h(p_i),\; i \in \{1, ..., n\}$$

show that $\{[\mu_1, ..., \mu_n]\} = \{[1, ..., 1]\} \in P$-Seq $\big(E(M_0)\big)$.

The proof of the main Theorem:

Let M be a generalized flag manifold for which the Conjecture C holds. Since M is a formal space we can find for every $N \in G(M)$ a rational equivalence

$f(N): N \to M$. Let $P(M)$ denote the set of primes which appear in any of the orders of

$$\ker\big(f(N)_*: H_*(N; \mathbf{Z}) \to H_*(M; \mathbf{Z})\big)$$

or coker $f(N)_*$, N ranging over $G(M)$. The set $P(M)$ is finite, since each ker $f(N)_*$ and coker $f(N)_*$ is finite and since $G(M)$ is a finite set by [W]. Consider now the map

$$\theta: G(M) \to P\text{-Seq } E(M_0)$$

with respect to this finite set of primes $P(M) = P$. Since P is finite,

$$P\text{-Seq}\big(E(M_0)\big)$$

consists of only one element (Lemma 2.2). It remains to show that

$$\theta^{-1}\big(\theta(M)\big) = \{M\}.$$

For this we apply Lemma 1.3. Note that $N \in G(M)$ implies $N_{\bar{P}} \simeq M_{\bar{P}}$ since $f(N): N \to M$ is a $\bar{P}$-equivalence. Moreover, the condition b) of 1.3 is satisfied in view of Lemma 2.1 b). Therefore we conclude that $G(M) = \{[M]\}$ and the proof is completed.

Note added in proof. Since this paper went to press, we have been informed that Conjecture C has been proved for the case $k = 2, n_1 = n_2$, by M. Hoffman: "Cohomology endomorphisms of complex flag manifolds", Ph.D. dissertation, MIT 1981. As a consequence, it follows that all complex Grassmann manifolds are generically rigid.