3. Classification of real normed division algebras

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for some complex λ. Suppose to the contrary that $x - \lambda e \neq 0$ for all λ in **C**. Since A is a division algebra, it follows that $x - \lambda e$ is invertible for all λ, i.e., $(x - \lambda e)^{-1}$ exists. Let $x(\lambda) = (x - \lambda e)^{-1}$. By the Hahn-Banach theorem there is a bounded linear functional L on A such that $L(x^{-1}) = 1$. Define $g : \mathbf{C} \to \mathbf{C}$ by $g(\lambda) = L(x(\lambda))$; then $g(0) = 1$. Moreover, g is an entire function. Indeed, since $x(\lambda) - x(\mu) = (\lambda - \mu)\, x(\lambda)\, x(\mu)$ for λ, μ in **C**, it follows that

$$\lim_{\lambda \to \mu} \frac{g(\lambda) - g(\mu)}{\lambda - \mu} = \lim_{\lambda \to \mu} L\left(x(\lambda)\, x(\mu)\right) = L\left(x(\mu)^2\right).$$

Further $|g(\lambda)| \leqslant ||L||\ ||x(\lambda)||$ and since $x(\lambda) \to 0$ as $|\lambda| \to \infty$, $g(\lambda) \to 0$. By Liouville's theorem the bounded entire function g is constant; hence $g \equiv 0$. This is a contradiction since $g(0) = 1$, and the proof is complete.

The *spectrum* of an element x of a complex algebra with identity e is the set $\sigma(x) = \{\lambda \in \mathbf{C} : x - \lambda e \text{ is singular}\}$, so Gelfand's proof can be viewed as a demonstration that the spectrum of any element of a complex normed algebra with identity is nonempty. This fact together with the application of Liouville's theorem forms a continuous thread running through the generalizations and related results presented in this paper.

3. Classification of real normed division algebras

Although it does not appear to be widely known, Mazur's original paper on normed division algebras [57] considers *only* the case of algebras over **R**. If a real division algebra is also finite-dimensional, the classical theorem of Frobenius classifies it as **R**, **C** ,or **H**. Mazur demonstrated finite-dimensionality in two steps: first he used a rather lengthy argument involving analytic function theory to show that it cannot contain a subalgebra isomorphic to the rational functions in one indeterminate with real coefficients. He then quoted an algebraic theorem to the effect that every real infinite-dimensional division algebra must contain such a subalgebra. The details of the first step may now be found in W. Żelazko's book [109, pp. 18-22].

F. F. Bonsall and J. Duncan [30] have given a more direct and self-contained proof of Mazur's theorem, which relies on precisely the same analytic fact as Gelfand's proof of the complex version; namely that every element of a complex normed algebra with identity has nonempty spectrum. They modify a standard proof of Frobenius' theorem (*vid.* Pontrjagin

[68, pp. 158-163]) by using the nonemptiness of the spectrum instead of finite-dimensionality to obtain the first major step. The remainder of the proof, though long, consists of entirely elementary algebraic verifications. We shall not reproduce this proof here.

If A is an algebra over **R**, its complexification $A_{\mathbf{C}}$ is analogous to the construction of **C** from **R**. (We may think of $A_{\mathbf{C}}$ as $A+iA$.) $A_{\mathbf{C}}$ will have a unit if and only if A does. Moreover if A is a normed algebra, the norm may be extended to $A_{\mathbf{C}}$ in a standard fashion (*vid.* Rickart [73, pp. 8-9]) so that the extension is complete whenever the original norm on A is. The spectrum $\sigma(x)$ of an element x in A is defined to be its spectrum in $A_{\mathbf{C}}$. Thus if A has a unit e, $\alpha + i\beta \in \sigma(x)$ if and only if the element $(\alpha+i\beta)(e, 0) - (x, 0)$ is singular in $A_{\mathbf{C}}$. Again by analogy with the complex numbers it is immediate that if a and b are *commuting* elements of A, (a, b) is invertible in $A_{\mathbf{C}}$ if and only if $a^2 + b^2$ is invertible in A. Thus $\alpha + i\beta \in \sigma(x)$ if and only if $(\alpha-x)^2 + \beta^2$ is singular in A.

4. Norm conditions and topological divisors of zero

In his original paper Mazur [57] also announced a companion theorem.

THEOREM 4.1. [Mazur]. *A real normed algebra A satisfying $\|xy\| = \|x\| \, \|y\|$ is isomorphic to* **R**, **C**, *or* **H**.

It is particularly worth noting that Theorem 4.1 (as stated in Mazurs' paper) carries no assumption that A has an identity element. Our goal in this section is to prove a generalization of this theorem due to Irving Kaplansky [53]. Its formulation depends on the concept of a topological divisor of zero in a normed algebra introduced by Shilov in 1940 [77]. An element x of a normed algebra is said to be a *topological divisor of zero* (t.d.z.) if there is a sequence y_n, $\|y_n\| = 1$, such that $xy_n \to 0$ or $y_n x \to 0$. Kaplansky's result is then:

THEOREM 4.2. [Kaplansky]. *If A is a real normed algebra having no nonzero topological divisors of zero, then A is isomorphic to* **R**, **C**, *or* **H**.

The development of the proof below closely follows Kaplansky's line of reasoning except for changes made to avoid the use of algebraic results not established here and instead to take advantage of Theorem 2.1. Mazur's original proof of Theorem 4.1 used algebraic results analogous to some