

# 3. The quadratic monoid

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **26 (1980)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

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that of product [2]) is no greater than that of  $T$ , to within a constant factor. However we have been unable to show the converse.

*Open Problem 1.* Is there an  $O(n^2)$  matrix-based algorithm for the  $T'$ -closure?

### 3. THE QUADRATIC MONOID

To satisfy our curiosity we investigated the monoid generated by the composition of closures corresponding to polynomials of degree at most two. For any set of transformations  $E$  let  $M_E$  be the monoid generated by compositions of elements of  $E$ . For any polynomial  $P(X, \bar{X})$ , define  $Z_P : A \rightarrow (\mu X. X \geq A \vee P(X, \bar{X}))$  and then

$$\Pi_r = \{ Z_P \mid \deg(P) \leq r \}.$$

THEOREM 3.  $M_{\Pi_2} = M_{\{R, S, Q, Q', T, T'\}}$  and the monoid is finite.

*Proof.* The equality follows from the finiteness since

$$\begin{aligned} Z_{P_1 \vee P_2} &= \bigvee_m (Z_{P_1} \cdot Z_{P_2})^m \\ &\in M_{\{Z_{P_1}, Z_{P_2}\}} \quad \text{if this is finite.} \end{aligned}$$

$M_{\{R, S, Q, Q', T, T'\}}$  is examined explicitly below and is found to contain exactly fifty elements. □

We write  $A$  for the monoid identity given by  $A^A = A$  and  $[Z_1, \dots, Z_k]$  for the closure  $\bigvee_m (Z_1 \vee \dots \vee Z_k)^m$ . Together with the obvious idempotencies of closures we have the following sufficient defining relations.

$$\begin{aligned} W &\stackrel{\text{def}}{=} [S, Q, Q', T, T'] \\ &= QQ' = Q'Q = QT' = Q'T' = SQ = SQ' = ST = ST' \\ V &\stackrel{\text{def}}{=} [R, S, Q, Q', T, T'] = WR = RQ = RQ' = RT' \\ &QT = [Q, T] \quad Q'T = [Q', T] \\ T'Q &= T'TQ = T'QT \quad T'Q' = T'TQ' = T'Q'T \\ TT' &= T'T * \stackrel{\text{def}}{=} RT = TR \quad RS = SR \end{aligned}$$

The closures in the monoid are

$$\begin{aligned} V &: A^V = (\bar{A} \vee A)^* \\ W &: A^W = (\bar{A} \vee A)^+ \end{aligned}$$

$$[Q, T] : A^{[Q, T]} = A^V \cdot A$$

$$[Q', T] : A^{[Q', T]} = A \cdot A^V$$

$$[T, T'] : A^{[T, T']} = A \vee A^V \cdot (AA \vee \bar{A}\bar{A}) \cdot A^V$$

\*,  $[R, S], R, S, Q, Q', T, T'$  and  $A$ .

The monoid can be counted after expressing its elements in a canonical form by the following rules.

- (i) Using  $RS = SR, RT = TR, RQ = RQ' = RT' = QQ'R$ , we can bring any occurrence of  $R$  to the end of the product
- (ii) Using  $SQ = SQ' = ST = ST' = QQ'$ , we can assume that any  $S$  occurs at the end of the rest of the product
- (iii)  $QT' = Q'T' = T'QQ'$  and  $TT' = T'T$  allow us to bring any  $T'$  to the front of the remainder.
- (iv) The elements generated by  $Q, Q', T$  are found to be

$$A, Q, Q', T, TQ, TQ', QT, Q'T, W$$

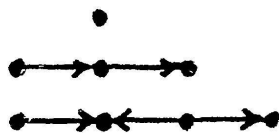
Prefixing these with  $T'$  yields only 4 new elements

$$T', T'Q, T'Q', T'T$$

- (v) The 50 elements of the monoid are given by

$$\{A, Q, Q', T, T', TQ, TQ', QT, Q'T, T'Q, T'Q', T'T\} \cdot \{A, R, S, SR\} \cup \{W, V\}$$

These elements are distinguishable by their effect on the graph



The “fast” monoid generated by the  $O(n^2)$  operations  $R, S, Q, Q'$  has only 14 elements

$$\{A, Q, Q'\} \cdot \{A, R, S, SR\} \cup \{W, V\}$$

Of computational interest are the relations

$$RQ = V \quad \text{and} \quad QQ' = SQ = W$$

which yield efficient ways to compute these common closures. Note that in some contexts the  $Q'$  closure may be more rapid to compute than  $S$ .

We illustrate some of the proofs for the results above.

THEOREM 4.

CANCELLATION LEMMA. For all  $A$ ,  $A\bar{A}A \geq A$ .

*Proof.*  $(A\bar{A}A)_{ij} \geq A_{ij}\bar{A}_{ji}A_{ij} = A_{ij}A_{ij}A_{ij} \geq A_{ij}$  □

- (i)  $RQ = V$
- (ii)  $QQ' = W$
- (iii)  $[Q, T] = QT$  and  $A^{QT} = A^V \cdot A$

*Proof.*

- (i) If we show that  $RQ \geq S$  the result follows easily. But

$$A^{RQ} \geq (I \vee A)^Q \geq A \vee \bar{A}I = A \vee \bar{A} = A^S$$

- (ii) Again the only non-trivial part is that  $QQ' \geq S$

$$A^{QQ'} \geq (A \vee \bar{A}A)^{Q'} \geq A \vee \bar{A}A \cdot \bar{A} \geq A^S$$

by the Cancellation Lemma.

- (iii) By inspection,  $A^{[Q, T]} \leq A^V \cdot A$

However,

$$A^V \cdot A = A^* \cdot \bar{A}A^V A \vee A^* \cdot A \leq (A^Q)^T \leq A^{[Q, T]} \quad \square$$

One of the harder results to prove is that  $TT' = T'T$ . We leave it as an exercise for the reader.

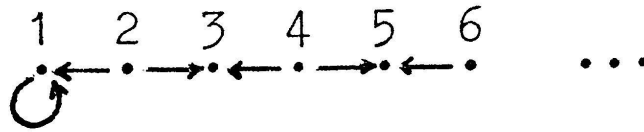
We have found that each mapping in  $\Pi_2$  is defined by a regular set over  $\{A, \bar{A}\}$ , however in  $\Pi_3$  there are «non-regular» mappings, e.g.  $Z_{XXX} \vee XXX$ .

The finiteness of  $M_{\Pi_2}$  does not persist for large  $r$ . We show by the example below that  $M_{\Pi_4}$  is infinite.

*Open Problem 2.* Is  $M_{\Pi_3}$  finite?

*Example.* Let  $J = Z_{XXXX}$ ,  $K = Z_{XXXX}$

$Z_{XXXX} \vee_{XXXX} \notin M_{\{J, K\}}$  and so  $M_{\{J, K\}}$  is infinite, since for the infinite graph shown below:



$(JK)^m$  adds all edges  $\langle i, j \rangle$  with  $i, j \leq 2m$   
 and  $(JK)^m J$  adds all edges  $\langle i, j \rangle$  with  $i, j \leq 2m + 1$   
 Therefore  $J, JK, JKJ, \dots$  are all distinct.

#### 4. GENERALIZED ALGORITHM FOR POWER-GROUP ALGEBRAS

To elucidate the correctness of the algorithm and to encompass some more general applications we need to generalize from the  $\{0, 1\}$  Boolean algebra to a slightly richer structure. The *power-group algebra*  $P(G)$  is a structure defined from an arbitrary group  $G$ . The elements of  $P(G)$  are the subsets of  $G$ ; the operations we require are *union* ( $\cup$ ), *complex product*:

$$ab = \{gh \mid g \in a, h \in b\} \text{ for } a, b \subseteq G$$

and *converse*:

$$\bar{a} = \{g^{-1} \mid g \in a\}$$

$P(G)$  is a monoid with respect to product with identity  $\lambda = \{\text{identity}_G\}$ . As before we shall be considering matrices over the structure, with matrix product and union defined in the obvious way from product and union in  $P(G)$ , and matrix *converse* defined by

$$(\bar{A})_{ij} = \bar{A}_{ji}$$

The key properties of power-group algebras which are needed are given below

LEMMA. Let  $a, b$  be elements and  $A, B$  matrices

- (i)  $\bar{\bar{a}} = a; \bar{\bar{A}} = A$
- (ii)  $\overline{ab} = \bar{b}\bar{a}; \overline{AB} = \bar{B}\bar{A}$
- (iii) if  $a \neq \emptyset$  then  $a\bar{a} \supseteq \lambda; A\bar{A}A \supseteq A$