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Autor(en): **Diamond, Harold G. / Erdös, Paul**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **26 (1980)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-51076

# ON SHARP ELEMENTARY PRIME NUMBER ESTIMATES

by Harold G. DIAMOND * and Paul ERDÖS

*Dedicated to the memory of László Kalmár*

## 1. INTRODUCTION

The first estimates of the true magnitude of the prime counting function $\pi(x) = \# \{p \leqslant x : p \text{ prime}\}$ were made by Chebyshev [1] in the mid nineteenth century. By an ingenious argument he proved that as $x \to \infty$

$$.92129 \ldots \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \leqslant \pi(x) \leqslant 1.10555 \ldots \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

These bounds were somewhat improved by J.-J. Sylvester and others, but nearly a half a century passed before Hadamard and de la Vallée Poussin independently succeeded in proving that $\pi(x) / \{x/\log x\} \to 1$ as $x \to \infty$. This is the famous prime number theorem (P.N.T.). Interesting accounts of the foregoing may be found in the books of Ingham [2], Landau [3], and Mathews [4].

Here we consider the following hypothetical question. Could Chebyshev in principle have achieved sharper bounds? We answer this question in the affirmative in the following

THEOREM. *Let* $\varepsilon > 0$ *be given. There exists a positive integer* $T = T(\varepsilon)$ *such that knowledge of the values of the Mobius* $\mu$ *function on the interval* $[1, T)$ *yields the estimate*

$$\limsup_{x \to \infty} \left| \pi(x)/(x/\log x) - 1 \right| < \varepsilon . \tag{1}$$

* Research supported in part by a grant from the National Science Foundation.

The main step in the proof is to reduce the estimate in (1) for a given value of $\varepsilon$ to an inequality of the form

$$|G_T(y)| \leqslant T/K \qquad (1 \leqslant y \leqslant KT),$$

where $T$ and $K$ depend only on $\varepsilon$ and $G_T$ is an auxiliary function defined in §2, depending only on the Möbius function $\mu(n)$ for $n < T$. Our proof of this inequality for general $T$ and $K$ (Lemma 2) uses results known to be equivalent to the P.N.T., and so our method does not lead to a new proof of the P.N.T. However, one can in principle verify the estimate of $G_T(y)$ numerically, which leads to an explicit estimate of the type (1). We use such numerical estimates in §3 to obtain bounds for $\pi(x)$ which are sharper than those of Chebyshev.

Our theorem has a long and curious history. It was first found in 1937 by Erdös and László Kalmár and independently, and at about the same time, by J. Barkley Rosser. Erdös and Kalmár decided not to publish it when they learned that Rosser had a version of the theorem for primes in arithmetic progression and had already submitted a manuscript for publication. However, because of various difficulties, Rosser's article never got into print, and the theorem lived only by word of mouth. We have reconstructed a proof which we give below.

2. Proof of the theorem. We denote by 1, $\mu$, $L$, $e_k$, and $\Lambda$ the following arithmetic functions:

$$1(n) = 1,$$

$$\mu(n) = \text{Möbius' function},$$

$$L(n) = \log n,$$

$$e_k(n) = \begin{cases} 1, n = k, \\ 0, n \neq k, \end{cases} \qquad \text{(here } k \text{ is a fixed positive integer)}$$

$$\Lambda(n) = \begin{cases} \log p, n = p^\alpha \\ 0, \quad \text{otherwise}. \end{cases} \qquad \text{(von Mangoldt's function)}$$

Also, let

$$\psi(x) = \sum_{n \leqq x} \Lambda(n) = \sum_{p^\alpha \leqq x} \log p .$$

It suffices to establish

$$\limsup_{x \to \infty} \left| \psi(x)/x - 1 \right| < \varepsilon \tag{2}$$

instead of (1) because of the familiar inequalities

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p \leqslant \sum_{p \leq x} \log x = \pi(x) \log x\,,$$

$$\pi(x) \leqslant \sum_{n \leq x} \frac{\Lambda(n)}{\log n} = \frac{\psi(x)}{\log x} + \int_1^x \frac{\psi(t)}{t \log^2 t}\, dt \leqslant \frac{\psi(x)}{\log x} + \frac{Bx}{\log^2 x}\,.$$

(The last inequality uses the Chebyshev bound $\psi(x) = O(x)$).

For $f$ and $g$ arithmetic functions define the multiplicative convolution of $f$ and $g$ by

$$f * g(n) = \sum_{ij=n} f(i)\, g(j)\,.$$

Convolution is an associative operation, and $e_1$ is the identity element. Familiar relations conveniently expressed in convolution terms are the Möbius identity $1 * \mu = e_1$ and the Chebyshev identity $\Lambda * 1 = L$.

One can determine $\Lambda$ from the last identity by convolving both sides by $\mu$; however, it is then difficult to handle $L * \mu$ effectively because of the irregularity of $\mu$. For $T$ a positive integer set

$$\mu_T(n) = \begin{cases} \mu(n), & 1 \leqslant n < T, \\ -T \displaystyle\sum_{i<T} \frac{\mu(i)}{i}, & n = T, \\ 0, & n > T. \end{cases}$$

We shall take $\mu_T$ as a "finite approximation" to $\mu$.

We first note a few properties of $\mu_T$. By construction, $\Sigma\, \mu_T(n)/n = 0$. For $x \geqslant 1$, the relation

$$\sum_{s \leq x} \frac{x}{s} \mu(s) = \sum_{s \leq x} \left[\frac{x}{s}\right] \mu(s) + \sum_{s \leq x} \left(\frac{x}{s} - \left[\frac{x}{s}\right]\right) \mu(s)$$

$$= 1 + \sum_{s \leq x} \left\{\frac{x}{s}\right\} \mu(s)$$

implies that $\mu_T(T) = O(T)$. Also, for $x < T$ we have

$$\sum_{n \leq x} 1 * \mu_T(n) = \sum_{n \leq x} 1 * \mu(n) = \sum_{n \leq x} e_1(n) = 1\,.$$

(Chebyshev used $f = e_1 - e_2 - e_3 - e_5 + e_{30}$ as an approximation to $\mu$ in his argument. This function has the properties that $\sum f(n)/n = 0$,

$$0 \leqslant F(x) \overset{\text{def}}{=} \sum_{n \leqq x} 1 * f(n) \leqslant 1 \qquad (x \geqslant 1),$$

and $F(x) = 1$ for $1 \leqslant x < 6$.)

Our starting point is the identity

$$\sum_{n \leqq x} \Lambda * 1 * \mu_T(n) = \sum_{n \leqq x} L * \mu_T(n). \tag{3}$$

We shall show that the left side of (3) is nearly $\psi(x)$ and right side is nearly $x$. We begin by estimating the right side of (3) unconditionally.

Lemma 1. *Given* $\varepsilon > 0$ *there exists an unbounded sequence of integers* $T$ *such that*

$$\Big| \sum_{n \leqq x} L * \mu_T(n) - x \Big| < \varepsilon x \qquad (x \geqslant x(T)).$$

*Proof.* Summation by parts shows that

$$\sum_{n \leqq y} \log n = y \log y - y + O(\log ey) \qquad (y \geqslant 1).$$

Thus

$$\begin{aligned}
\sum_{n \leqq x} L * \mu_T(n) &= \sum_{n \leqq x} \sum_{ij = n} \log i \, \mu_T(j) = \sum_{ij \leqq x} \log i \, \mu_T(j) \\
&= \sum_{j \leqq x} \Big( \sum_{i \leqq x/j} \log i \Big) \mu_T(j) \\
&= \sum_{j \leqq x} \left\{ \frac{x}{j} (\log x - \log j - 1) + O(\log ex/j) \right\} \mu_T(j) \\
&= (x \log x - x) \sum_{j \leqq x} \mu_T(j)/j - x \sum_{j \leqq x} \frac{\log j}{j} \mu_T(j) \\
&\quad + O\Big\{ (\log ex) \sum_{j \leqq x} |\mu_T(j)| \Big\}.
\end{aligned}$$

Suppose that $x \geqslant T$. Then the first sum is zero by the construction of $\mu_T$. The third term is $O(T \log ex)$.

To evaluate the second term set

$$m(u) = \sum_{i \leqq u} \mu(i)/i, \qquad m_1(y) = \int_1^y m(u) \, du/u.$$

The definition of $\mu_T$ and integration by parts give

$$-\sum_{j\leqq T}\frac{\log j}{j}\,\mu_T(j) = \sum_{j\leqq T}\log\frac{T}{j}\frac{\mu(j)}{j}$$

$$= \int_{1-}^{T}\log\frac{T}{u}\,dm(u) = m_1(T).$$

We claim that $m_1(T)\to 1$ for a sequence of $T$'s tending to infinity. Indeed, since $s\int_1^\infty u^{-s-1}\,du = 1$, an average of $m_1$ is provided by

$$s\int_1^\infty u^{-s-1}\,m_1(u)\,du = \frac{1}{s\zeta(s+1)}$$

(two integrations by parts), and the last expression approaches 1 as $s\to 0+$. Also, $m_1(T+a) - m_1(T)\to 0$ as $T\to\infty$, $0\leqslant a\leqslant 1$. Thus it is impossible that $m_1(n)$ be ultimately bounded away from 1 for integral values of $n$. □

To evaluate the left side of (3), set

$$g = g_T = 1 * \mu_T, \qquad G(x) = G_T(x) = \sum_{n\leqq x} g(n).$$

Then we have (by the "hyperbola method")

$$\sum_{n\leqq x}\Lambda * g(n) = \sum_{ij\leqq x}\Lambda(i)\,g(j)$$

$$= \sum_{j\leqq T-1}\psi(x/j)\,g(j) + \sum_{i\leqq x/(T-1)} G(x/i)\,\Lambda(i) - G(T-1)\,\psi\left(\frac{x}{T-1}\right)$$

$$= \mathrm{I} + \mathrm{II} - \mathrm{III},\ \text{say}.$$

Since $\mu_T(n) = \mu(n)$ for all $n < T$ we have $g(j) = 1 * \mu(j)$ for all $j < T$. Thus $\mathrm{I} = \psi(x)$ and $\mathrm{III} = \psi(x/(T-1)) = O(x/T)$.

We study $G$ to estimate II. For $y\geqslant T$ we have

$$G(y) = \sum_{n\leqq y} g(n) = \sum_{ij\leqq y} 1(i)\,\mu_T(j)$$

$$= \sum_{j\leqq y}[y/j]\,\mu_T(j) = \sum_{j\leqq T}\left(\frac{y}{j} - \left\{\frac{y}{j}\right\}\right)\mu_T(j).$$

Thus

$$G(y) = -\sum_{j<T}\{y/j\}\,\mu(j) + T\{y/T\}\sum_{j<T}\mu(j)/j. \tag{4}$$

The last sum is bounded, as we noted earlier, and so $G(y) = O(T)$ for all $y\geqslant T$.

For $K$ a large positive number, write

$$\sum_{i \leq x/(T-1)} G\left(\frac{x}{i}\right) \Lambda(i) = \sum_{i \leq x/TK} + \sum_{x/TK < i \leq x/(T-1)} G\left(\frac{x}{i}\right) \Lambda(i) .$$

The first sum is $O(x/K)$ by the preceding estimate and Chebyshev's bound $\psi(y) = O(y)$. We shall use the prime number theorem to estimate the size of $G(x/i)$ in the last sum.

Lemma 2. *Let* $K > 0$. *There exists a number* $T_0 = T_0(K)$ *such that if* $T \geqslant T_0$ *then* $|G_T(y)| < T/K$ *holds for all* $y \leqslant TK$.

*Proof.* We may suppose that $y \geqslant T$, for $G(y) = 1$ for $1 \leqslant y < T$. We estimate $G$ using (4). A result "equivalent" to the P.N.T. is the estimate

$$\sum_{j<T} \mu(j)/j \to 0 \qquad (T \to \infty) .$$

Thus, if $T$ is large enough we have

$$\left| T\left\{\frac{y}{T}\right\} \sum_{j<T} \mu(j)/j \right| < T/3K .$$

The remaining sum in (4) is

$$\sum_{j<T} \left\{\frac{y}{j}\right\} \mu(j) = \sum_{j \leq T/3K} + \sum_{T/3K<j<T} \left\{\frac{y}{j}\right\} \mu(j) .$$

The first sum on the right is bounded by $T/3K$ in modulus.

We estimate the last sum by breaking its summation range into subintervals $a < j \leqslant b$ on which $[y/j]$ is constant. The number of such ranges cannot exceed the maximal value of $[y/j]$, i.e. $TK/(T/3K) = 3K^2$. On each interval we sum by parts and use the monotonicity of $\{y/j\}$ and the estimate

$$M(z) = \sum_{n \leq z} \mu(n) = o(z) \qquad (z \to \infty),$$

which follows from the P.N.T. We obtain

$$\sum_{a<j\leq b} \left\{\frac{y}{j}\right\} \mu(j) = \left\{\frac{y}{b}\right\} M(b) - \left\{\frac{y}{a+1}\right\} M(a)$$

$$+ \sum_{a<j<b} \left(\left\{\frac{y}{j}\right\} - \left\{\frac{y}{j+1}\right\}\right) M(j) = o(b) \leqslant T/9K^3$$

provided that $T$ is sufficiently large. Thus

$$\left| \sum_{j<T} \left\{ \frac{y}{j} \right\} \mu(j) \right| < \frac{2T}{3K},$$

and $| G(y) | < T/K$ for all $y \leqslant TK$. □

Returning to the estimate of II we have

$$\left| \sum_{i \leq x/(T-1)} G\left(\frac{x}{i}\right) \Lambda(i) \right| \leqslant O\left(\frac{x}{K}\right) + \sum_{x/TK<i\leq x/(T-1)} \frac{T}{K} \Lambda(i) = O\left(\frac{x}{K}\right),$$

provided that $T$ is sufficiently large.

Combining all estimates involving (3) we obtain finally

$$\psi(x) + O\left(\frac{x}{T}\right) + O\left(\frac{x}{K}\right) = x + O(\varepsilon x),$$

where $\varepsilon$, $1/T$ and $1/K$ can all be taken as small as we wish. □

3. Calculations. We conclude by applying our method to obtain bounds for $\psi(x)/x$ sharper than those of Chebyshev (though not as good as the bounds of Sylvester).

Returning to (3), we have

$$\sum_{n<x} L^* \mu_T(n) = x m_1(T) + O(T \log ex),$$

and direct calculation shows that $m_1(T)$ is close to 1 for a sequence of $T$'s. For example, we have

| $T$ | 3 | 5 | 7 | 11 | 32 | 152 |
|---|---|---|---|---|---|---|
| $m_1(T) - 1$ | −.104 | −.019 | −.0045 | −.00072 | −.000030 | −.00000037 |

We write the left side of (3) as

$$\psi(x) - \psi\left(\frac{x}{T-1}\right) + \sum_{i \leq \frac{x}{TK}} + \sum_{\frac{x}{TK} < i \leq \frac{x}{T-1}} G_T\left(\frac{x}{i}\right) \Lambda(i).$$

In the first sum we use the bound $|G_T(x/i)| \leqslant 3T/2$, which is easily deduced from (4); for the second sum we calculate $G_T(y)$ directly for $T - 1 \leqslant y < TK$, and let $U = U(T, K)$ and $L = L(T, K)$ denote the upper and lower bounds for $G_T$ on this interval.

Combining the estimates we obtain

$$(5)\quad \psi(x) - \psi\left(\frac{x}{T-1}\right) - \frac{3}{2}T\psi\left(\frac{x}{TK}\right) + L\left\{\psi\left(\frac{x}{T-1}\right) - \psi\left(\frac{x}{TK}\right)\right\} \leqslant x m_1(T) + O(T \log ex),$$

$$(6)\quad \psi(x) - \psi\left(\frac{x}{T-1}\right) + \frac{3}{2}T\psi\left(\frac{x}{TK}\right) + U\left\{\psi\left(\frac{x}{T-1}\right) - \psi\left(\frac{x}{TK}\right)\right\} \geqslant x m_1(T) + O(T \log ex).$$

We give an upper estimate of $\psi(x)/x$ using (5) with $T = 100$, $TK = 8911$, $L \geqslant -4.9054$, $m_1(100) \leqslant 1.00104$, and Chebyshev's bound $\limsup \psi(x)/x < 1.1056$. We find that $\limsup \psi(x)/x < 1.085$. We estimate $\psi(x)/x$ from below by using (6) with $T = 101$, $TK = 17749$, $U \leqslant 7.2930$, $m_1(101) \geqslant 1.00134$ and the preceding upper estimate of $\psi(x)/x$. We find that $\liminf \psi(x)/x > .924$.

Might Chebyshev have improved his bounds for $\psi(x)/x$ if he had used this method? We must report that that is quite unlikely, because considerable calculation was needed to obtain the modest improvement we have achieved.

NOTE ADDED IN PROOF. Diamond and Kevin Mc Curley have found another sharp elementary estimation method. Their article "Constructive elementary estimates for $M(x)$" will appear in *Number Theory — Proceedings of a conference held at Temple University, May 1980*, Lectures Notes in Math., Springer-Verlag, Berlin.

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*(Reçu le 20 janvier 1980)*

Harold G. Diamond

Department of Mathematics
University of Illinois
Urbana, Illinois 61801

Paul Erdös

Hungarian Academy of Science
Reáltanoda U. 13-15
Budapest

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