12. The quadratic form

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12. The quadratic form

Let $f(z_0, \ldots, z_n)$ be a germ with $f(\mathbf{0}) = 0$ and an isolated critical point at $\mathbf{0}$ (that is, a germ in $\mathscr{F}$). There is an $\varepsilon > 0$ such that $f^{-1}(0)$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leqslant \varepsilon$. For suitably small $\delta > 0$, $f^{-1}(\delta')$ intersects the closed disk D_ε^{2n+2} of radius ε transversally for all $|\delta'| \leqslant \delta$. Let

$$F = f^{-1}(\delta) \cap D_\varepsilon^{2n+2}$$

be the *Milnor fiber* of f [Milnor 1]. The set F is a smooth real $2n$-manifold with boundary whose diffeomorphism type is independent of the choice of ε and δ. Furthermore, F is $(n-1)$-connected, and the Milnor number μ of §7 is the rank of $H_n(F)$. The Milnor number is zero if and only if the germ f has a regular point at $\mathbf{0}$ [Milnor 1, Corollary 7.3]. The *intersection pairing* $(\ ,\ )$ of F is the integral bilinear form $H_n(F) \times H_n(F) \to \mathbf{Z}$ defined by sending (x, y) to $(x' \cup y')\,[F]$, where x' and y' in $H^n(F, \partial F)$ are Lefschetz duals to x and y, and $[F]$ in $H_{2n}(F, \partial F)$ is the orientation class of F given by the underlying complex structure. The intersection pairing is symmetric if n is even, and skew symmetric if n is odd. For example, the germ $f(z_0, \ldots, z_n) = z_0^2 + \ldots + z_n^2$ has $H_n(F)$ a free cyclic group with generator e, and $(e, e) = 2(-1)^{n/2}$ or 0 according as n is even or odd. There are many methods of computing the intersection pairing in special cases.

By a tensor product theorem [Gabrielov 1; Sakamoto], the Milnor numbers of $f(z_0, \ldots, z_n)$ and $f(z_0, \ldots, z_n) + z_{n+1}^2 + \ldots + z_m^2$ are equal. The *quadratic form* of $f(z_0, \ldots, z_n)$ is defined to be the intersection pairing of the germ $f(z_0, \ldots, z_n) + z_{n+1}^2 + \ldots + z_m^2$ where $m \equiv 2 \pmod 4$. This is independent of the choice of m. For example, if $n \equiv 0 \pmod 4$ then the quadratic form of f is the negative of its intersection pairing; all this follows from the tensor product theorem. See also [Kauffmann and Neumann].

A germ f *topologically degenerates* to a germ g if there is an $\eta > 0$ and a family h_t of germs for $\{t \in \mathbf{C} : |t| < 2\eta\}$ with $h_\eta \sim f$, $h_0 \sim g$, and h_t of constant Milnor number for $t \neq 0$. Compare [Lê and Ramanujam]. Clearly right degeneracy implies topological degeneracy.

Lemma 12.1 [Tjurina 1, Theorem 1]. If f topologically degenerates to g, then there is an injection of $H_n(F_f)$ into $H_n(F_g)$ (where F_f is the Milnor fiber of f, and F_g is the Milnor fiber of g), and this injection preserves the intersection pairing. In particular, if g topologically degenerates to f as well, then the intersection pairings of f and g are isomorphic.

Characterization B5. The quadratic form of f is negative definite.

The equivalence of Characterizations B1 and B5 is proved in [Tjurina 1]. By explicit computation the quadratic forms of the germs in Table 2a are shown to be negative definite, and those of Table 2b are shown to be negative semi-definite. (In fact, the quadratic form of a germ in Table 2a is isomorphic to the intersection pairing of its minimal resolution, and the quadratic form of a germ of type $\tilde{E}_k$ in Table 2b is isomorphic to the quadratic form of E_k plus a two-dimensional zero form.) The result then follows from Proposition 10.1 and Lemma 12.1. When $n = 2$, the Milnor fiber F is in fact diffeomorphic to the minimal resolution M of $f^{-1}(0)$, since the singularity of $f^{-1}(0)$ is an absolutely isolated double point [Brieskorn 1, Theorem 4; Tjurina 1, Lemma 1].

When $n = 2$, the equivalence of Characterizations A2 and B5 follows from the following result [Durfee 2, Proposition 3.1].

THEOREM 12.2. *Twice the geometric genus p of $f^{-1}(0)$ equals the number of positive plus the number of zero diagonal elements in a diagonalization of the intersection pairing over the real numbers.*

The classification of germs according to signature of the quadratic form has been extended in [Arnold 3]; see also [Durfee 2, Proposition 3.3].

13. Nearby Morse functions

A *deformation* of a germ $f \in \mathscr{F}$ is a germ $g: \mathbf{C}^{n+1} \times \mathbf{C} \to \mathbf{C}$ with $g(z, 0) = f(z)$. Choose ε and δ for f as in §11. Then choose $\eta > 0$ such that for all $|t| < \eta$ and $|\delta'| \leqslant \delta$, the set $\{z \in \mathbf{C}^{n+1} : g(z, t) = \delta'\}$ intersects S_ε^{2n+1} transversally and the critical values of $g(z, t)$ for fixed t are less than δ in absolute value. A germ $\bar{f}$ is a *nearby Morse function* to f if $\bar{f}$ has only non-degenerate critical points in D_ε^{2n+2} and there is a deformation g and a t_0 with $|t_0| < \eta$ such that $\bar{f}(z) = g(z, t_0)$.

Characterization B6. There is a nearby Morse function to f with one or two critical values.

In fact, the nearby Morse function has one critical value if and only if f is right equivalent to A_2, since the quadratic form diagram is connected (§14). This surprising characterization is in [A'Campo 2II], where it is shown that Characterization B1 implies B6, and B6 implies B7 below.