

# 12. The quadratic form

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

## 12. THE QUADRATIC FORM

Let  $f(z_0, \dots, z_n)$  be a germ with  $f(\mathbf{0}) = 0$  and an isolated critical point at  $\mathbf{0}$  (that is, a germ in  $\mathcal{F}$ ). There is an  $\varepsilon > 0$  such that  $f^{-1}(0)$  intersects all spheres of radius  $\varepsilon'$  about  $\mathbf{0}$  transversally for  $0 < \varepsilon' \leq \varepsilon$ . For suitably small  $\delta > 0$ ,  $f^{-1}(\delta')$  intersects the closed disk  $D_\varepsilon^{2n+2}$  of radius  $\varepsilon$  transversally for all  $|\delta'| \leq \delta$ . Let

$$F = f^{-1}(\delta) \cap D_\varepsilon^{2n+2}$$

be the *Milnor fiber* of  $f$  [Milnor 1]. The set  $F$  is a smooth real  $2n$ -manifold with boundary whose diffeomorphism type is independent of the choice of  $\varepsilon$  and  $\delta$ . Furthermore,  $F$  is  $(n-1)$ -connected, and the Milnor number  $\mu$  of §7 is the rank of  $H_n(F)$ . The Milnor number is zero if and only if the germ  $f$  has a regular point at  $\mathbf{0}$  [Milnor 1, Corollary 7.3]. The *intersection pairing*  $(,)$  of  $F$  is the integral bilinear form  $H_n(F) \times H_n(F) \rightarrow \mathbf{Z}$  defined by sending  $(x, y)$  to  $(x' \cup y')$   $[F]$ , where  $x'$  and  $y'$  in  $H^n(F, \partial F)$  are Lefschetz duals to  $x$  and  $y$ , and  $[F]$  in  $H_{2n}(F, \partial F)$  is the orientation class of  $F$  given by the underlying complex structure. The intersection pairing is symmetric if  $n$  is even, and skew symmetric if  $n$  is odd. For example, the germ  $f(z_0, \dots, z_n) = z_0^2 + \dots + z_n^2$  has  $H_n(F)$  a free cyclic group with generator  $e$ , and  $(e, e) = 2(-1)^{n/2}$  or 0 according as  $n$  is even or odd. There are many methods of computing the intersection pairing in special cases.

By a tensor product theorem [Gabrielov 1; Sakamoto], the Milnor numbers of  $f(z_0, \dots, z_n)$  and  $f(z_0, \dots, z_n) + z_{n+1}^2 + \dots + z_m^2$  are equal. The *quadratic form* of  $f(z_0, \dots, z_n)$  is defined to be the intersection pairing of the germ  $f(z_0, \dots, z_n) + z_{n+1}^2 + \dots + z_m^2$  where  $m \equiv 2 \pmod{4}$ . This is independent of the choice of  $m$ . For example, if  $n \equiv 0 \pmod{4}$  then the quadratic form of  $f$  is the negative of its intersection pairing; all this follows from the tensor product theorem. See also [Kauffmann and Neumann].

A germ  $f$  *topologically degenerates* to a germ  $g$  if there is an  $\eta > 0$  and a family  $h_t$  of germs for  $\{t \in \mathbf{C} : |t| < 2\eta\}$  with  $h_\eta \sim f$ ,  $h_0 \sim g$ , and  $h_t$  of constant Milnor number for  $t \neq 0$ . Compare [Lê and Ramanujam]. Clearly right degeneracy implies topological degeneracy.

*Lemma 12.1* [Tjurina 1, Theorem 1]. If  $f$  topologically degenerates to  $g$ , then there is an injection of  $H_n(F_f)$  into  $H_n(F_g)$  (where  $F_f$  is the Milnor fiber of  $f$ , and  $F_g$  is the Milnor fiber of  $g$ ), and this injection preserves the intersection pairing. In particular, if  $g$  topologically degenerates to  $f$  as well, then the intersection pairings of  $f$  and  $g$  are isomorphic.

*Characterization B5.* The quadratic form of  $f$  is negative definite.

The equivalence of Characterizations B1 and B5 is proved in [Tjurina 1]. By explicit computation the quadratic forms of the germs in Table 2a are shown to be negative definite, and those of Table 2b are shown to be negative semi-definite. (In fact, the quadratic form of a germ in Table 2a is isomorphic to the intersection pairing of its minimal resolution, and the quadratic form of a germ of type  $\tilde{E}_k$  in Table 2b is isomorphic to the quadratic form of  $E_k$  plus a two-dimensional zero form.) The result then follows from Proposition 10.1 and Lemma 12.1. When  $n = 2$ , the Milnor fiber  $F$  is in fact diffeomorphic to the minimal resolution  $M$  of  $f^{-1}(0)$ , since the singularity of  $f^{-1}(0)$  is an absolutely isolated double point [Brieskorn 1, Theorem 4; Tjurina 1, Lemma 1].

When  $n = 2$ , the equivalence of Characterizations A2 and B5 follows from the following result [Durfee 2, Proposition 3.1].

**THEOREM 12.2.** *Twice the geometric genus  $p$  of  $f^{-1}(0)$  equals the number of positive plus the number of zero diagonal elements in a diagonalization of the intersection pairing over the real numbers.*

The classification of germs according to signature of the quadratic form has been extended in [Arnold 3]; see also [Durfee 2, Proposition 3.3].

### 13. NEARBY MORSE FUNCTIONS

A *deformation* of a germ  $f \in \mathcal{F}$  is a germ  $g: \mathbf{C}^{n+1} \times \mathbf{C} \rightarrow \mathbf{C}$  with  $g(z, 0) = f(z)$ . Choose  $\varepsilon$  and  $\delta$  for  $f$  as in §11. Then choose  $\eta > 0$  such that for all  $|t| < \eta$  and  $|\delta'| \leq \delta$ , the set  $\{z \in \mathbf{C}^{n+1}: g(z, t) = \delta'\}$  intersects  $S_\varepsilon^{2n+1}$  transversally and the critical values of  $g(z, t)$  for fixed  $t$  are less than  $\delta$  in absolute value. A germ  $\bar{f}$  is a *nearby Morse function* to  $f$  if  $\bar{f}$  has only non-degenerate critical points in  $D_\varepsilon^{2n+2}$  and there is a deformation  $g$  and a  $t_0$  with  $|t_0| < \eta$  such that  $\bar{f}(z) = g(z, t_0)$ .

*Characterization B6.* There is a nearby Morse function to  $f$  with one or two critical values.

In fact, the nearby Morse function has one critical value if and only if  $f$  is right equivalent to  $A_2$ , since the quadratic form diagram is connected (§14). This surprising characterization is in [A'Campo 2II], where it is shown that Characterization B1 implies B6, and B6 implies B7 below.