

## 2. Rational singularities

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

germs  $V$  and  $W$  embedded in  $\mathbf{C}^n$  at the origin are *isomorphic* if there is a germ of an analytic automorphism of  $\mathbf{C}^n$  fixing the origin and taking  $V$  to  $W$ .

*Characterization A1.* The analytic set  $f^{-1}(0)$  is isomorphic to the zero locus of one of the functions listed in column 1 of Table 1.

## 2. RATIONAL SINGULARITIES

A *resolution* of a germ of a normal surface singularity  $V$  as above is a complex analytic manifold  $M$  and an analytic map  $\pi: M \rightarrow V$  that is surjective and proper (compact fibers) such that its restriction to  $M - \pi^{-1}(\mathbf{v})$  is an analytic isomorphism, and  $M - \pi^{-1}(\mathbf{v})$  is dense in  $M$ . Resolutions exist, and can be computed with a certain amount of effort. The article [Lipman 2] contains a general discussion of resolutions, and [Laufer 1] and [Hirzebruch, Neumann, and Koh, §9] give a detailed method with examples.

Among all resolutions there is a *minimal resolution*  $\pi: M \rightarrow V$  that has the following universal mapping property: Given any other resolution  $\pi': M' \rightarrow V$ , there is a unique map  $\rho: M' \rightarrow M$  with  $\pi' = \pi \circ \rho$ .

The *geometric genus*  $p$  of  $V$  is the dimension of the complex vector space  $H^1(M, \mathcal{O}_M)$ , where  $M$  is any resolution of  $V$ , and  $\mathcal{O}_M$  is the sheaf of holomorphic functions on  $M$  [Artin; Wagreich 1, §1.4; Brieskorn 2; Laufer 2]. ( $V$  is assumed Stein.) This number is finite, and independent of the choice of resolution. It may alternately be defined as the dimension of the stalk at the origin of the sheaf  $R^1 \pi_* \mathcal{O}_M$  on  $V$ . The idea behind this definition is that  $M$  is a collection of “thickened” curves, and that the genus of a curve  $X$  is the dimension of  $H^1(X, \mathcal{O}_X)$ . For example,  $H^1(M, \mathcal{O}_M) = 0$  if  $M$  is the total space of a line bundle over a curve of genus zero. On the other hand,  $\dim H^1(M, \mathcal{O}_M) = k(k-1)(k-2)/6$  if  $M$  is a line bundle of Chern class  $-k$  over a curve of genus  $(k-1)(k-2)/2$  (the minimal resolution of  $f(x, y, z) = x^k + y^k + z^k$ ). In terms of  $V$  alone,  $p$  is the dimension of the space of holomorphic two-forms on  $V - \mathbf{v}$  divided by square-integrable forms [Laufer 2, Theorem 3.4]. Another formula for  $p$  in terms of topological invariants of the resolution  $M$  and the nearby fiber  $F$  (see §11) is given in [Laufer 6].

The analytic set  $V$  has a *rational* singularity if  $p = 0$ . A rational singularity embeds in codimension 1 if and only if it is a double point (its local ring is of multiplicity two) [Artin, Corollary 6].

*Characterization A2.* The singularity of  $f^{-1}(0)$  is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

### 3. EXCEPTIONAL SETS

Let  $V$  be as above, and let  $\pi: M \rightarrow V$  be a resolution of  $V$ . The *exceptional set*  $E = \pi^{-1}(\mathbf{v})$  is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves  $E_1, \dots, E_s$ . It is possible to arrange that the  $E_i$  are non-singular, the intersection of  $E_i$  and  $E_j$  is transverse for  $i \neq j$ , and no three  $E_i$  meet at a point. Such a resolution is called *good*. If, in addition, the intersection of  $E_i$  and  $E_j$  is empty or one point, the resolution is *very good*; this is possible to arrange as well.

Suppose that the resolution is good. Let  $E_i \cdot E_j$  equal the number of points of intersection of  $E_i$  and  $E_j$  if  $i \neq j$  (always a non-negative integer), or the first Chern class of the normal bundle to  $E_i$  evaluated on the orientation class of  $E_i$  if  $i = j$  (the self-intersection of  $E_i$ ). The matrix  $\{E_i \cdot E_j\}$  is called the *intersection matrix of the resolution*. It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves  $E = E_1 \cup \dots \cup E_s$  in a two-dimensional manifold  $M$  with negative definite intersection matrix  $\{E_i \cdot E_j\}$ , a theorem of Grauert says that the quotient space  $M/E$  has a normal complex structure and that the projection map  $M \rightarrow M/E$  is analytic [Laufer 1, p. 60].

*Characterization A3.* The minimal resolution of  $f^{-1}(0)$  is very good, and its exceptional set consists of curves of genus 0 and self-intersection  $-2$ .

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

- (i) Let  $M \rightarrow V$  be a resolution of a normal singularity  $V$  as above. There is a certain unique non-zero divisor  $Z = \sum n_i E_i$  on  $M$  with  $n_i \geq 0$  called the *fundamental cycle*, and it is shown that the singularity of  $V$  is rational if and only if the analytic Euler characteristic  $\chi(Z)$  of  $Z$  is 1 (that is, the arithmetic genus of  $Z$  is 0) [Artin, Theorem 3]. It is easy to see that the support of  $Z$  is the whole exceptional set of  $E$ .
- (ii) Any resolution of a rational singularity  $V$  is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].