2. Rational singularities

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

germs V and W embedded in $\mathbf{C}^n$ at the origin are *isomorphic* if there is a germ of an analytic automorphism of $\mathbf{C}^n$ fixing the origin and taking V to W.

Characterization A1. The analytic set $f^{-1}(0)$ is isomorphic to the zero locus of one of the functions listed in column 1 of Table 1.

2. Rational singularities

A *resolution* of a germ of a normal surface singularity V as above is a complex analytic manifold M and an analytic map $\pi: M \to V$ that is surjective and proper (compact fibers) such that its restriction to $M - \pi^{-1}(\mathbf{v})$ is an analytic isomorphism, and $M - \pi^{-1}(\mathbf{v})$ is dense in M. Resolutions exist, and can be computed with a certain amount of effort. The article [Lipman 2] contains a general discussion of resolutions, and [Laufer 1] and [Hirzebruch, Neumann, and Koh, §9] give a detailed method with examples.

Among all resolutions there is a *minimal resolution* $\pi: M \to V$ that has the following universal mapping property: Given any other resolution $\pi': M' \to V$, there is a unique map $\rho: M' \to M$ with $\pi' = \pi \circ \rho$.

The *geometric genus p* of V is the dimension of the complex vector space $H^1(M, \mathcal{O}_M)$, where M is any resolution of V, and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M [Artin; Wagreich 1, §1.4; Brieskorn 2; Laufer 2]. (V is assumed Stein.) This number is finite, and independent of the choice of resolution. It may alternately be defined as the dimension of the stalk at the origin of the sheaf $R^1 \pi_* \mathcal{O}_M$ on V. The idea behind this definition is that M is a collection of "thickened" curves, and that the genus of a curve X is the dimension of $H^1(X, \mathcal{O}_X)$. For example, $H^1(M, \mathcal{O}_M) = 0$ if M is the total space of a line bundle over a curve of genus zero. On the other hand, $\dim H^1(M, \mathcal{O}_M) = k(k-1)(k-2)/6$ if M is a line bundle of Chern class $-k$ over a curve of genus $(k-1)(k-2)/2$ (the minimal resolution of $f(x, y, z) = x^k + y^k + z^k$). In terms of V alone, p is the dimension of the space of holomorphic two-forms on $V - \mathbf{v}$ divided by square-integrable forms [Laufer 2, Theorem 3.4]. Another formula for p in terms of topological invariants of the resolution M and the nearby fiber F (see §11) is given in [Laufer 6].

The analytic set V has a *rational* singularity if $p = 0$. A rational singularity embeds in codimension 1 if and only if it is a double point (its local ring is of multiplicity two) [Artin, Corollary 6].

Characterization A2. The singularity of $f^{-1}(0)$ is rational.

Characterizations A1 and A2 will both be shown equivalent to Characterization A3.

3. Exceptional sets

Let V be as above, and let $\pi: M \to V$ be a resolution of V. The *exceptional set* $E = \pi^{-1}(\mathrm{v})$ is compact, one-dimensional, and connected, and hence is a union of irreducible complex curves $E_1, ..., E_s$. It is possible to arrange that the E_i are non-singular, the intersection of E_i and E_j is transverse for $i \neq j$, and no three E_i meet at a point. Such a resolution is called *good.* If, in addition, the intersection of E_i and E_j is empty or one point, the resolution is *very good*; this is possible to arrange as well.

Suppose that the resolution is good. Let $E_i \cdot E_j$ equal the number of points of intersection of E_i and E_j if $i \neq j$ (always a non-negative integer), or the first Chern class of the normal bundle to E_i evaluated on the orientation class of E_i if $i = j$ (the self-intersection of E_i). The matrix $\{E_i \cdot E_j\}$ is called the *intersection matrix of the resolution.* It is proved in [Du Val 2] (see also [Mumford; Laufer 1, p. 49]) that this matrix is negative definite. Conversely, given a collection of curves $E = E_1 \cup ... \cup E_s$ in a two-dimensional manifold M with negative definite intersection matrix $\{E_i \cdot E_j\}$, a theorem of Grauert says that the quotient space M/E has a normal complex structure and that the projection map $M \to M/E$ is analytic [Laufer 1, p. 60].

Characterization A3. The minimal resolution of $f^{-1}(0)$ is very good, and its exceptional set consists of curves of genus 0 and self-intersection -2.

The equivalence of Characterizations A2 and A3 is proved in [Du Val 1], and [Artin]. The following facts are needed:

(i) Let $M \to V$ be a resolution of a normal singularity V as above. There is a certain unique non-zero divisor $Z = \Sigma n_i E_i$ on M with $n_i \geqslant 0$ called the *fundamental cycle*, and it is shown that the singularity of V is rational if and only if the analytic Euler characteristic $\chi(Z)$ of Z is 1 (that is, the arithmetic genus of Z is 0) [Artin, Theorem 3]. It is easy to see that the support of Z is the whole exceptional set of E.

(ii) Any resolution of a rational singularity V is very good, and the curves in the exceptional set are of genus zero [Brieskorn 2, Lemma 1.3].