

## §2. Induced and coinduced acyclic maps

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§ 2. INDUCED AND COINDUCED ACYCLIC MAPS

(2.1) PROPOSITION. *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be two maps. If  $f$  and  $g$  are acyclic, then  $gf$  is acyclic. If  $f$  and  $gf$  are acyclic, then  $g$  is acyclic.*

*Proof.* Consider a local system  $L$  on  $Z$ , and using  $g^*L$  on  $Y$   $f^*g^*L = (gf)^*L$  on  $X$ , we apply (1.2) (b) to obtain the proposition.

(2.2) PROPOSITION. *Consider the following cartesian square where either  $f$  or  $g$  is a fibration.*

$$\begin{array}{ccc} Y' \times_Y Y & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

*If  $f$  is acyclic, then  $f'$  is acyclic.*

*Proof.* Since either  $f$  or  $g$  is a fibration, we can change the other to be a fibration, if necessary, without changing the homotopy type of any of the four spaces. Now the homotopy fibre  $F$  of  $f$  is the actual fiber and  $F$  is also the homotopy fibre of  $f'$ . Now apply (1.2) (a).

(2.3) PROPOSITION. *Consider the following cocartesian square where either  $f$  or  $g$  is a cofibration.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow g' \\ X' & \xrightarrow{f'} & X' \cup_X Y = Y' \end{array}$$

*If  $f$  is acyclic, then  $f'$  is acyclic.*

*Proof.* Since either  $f$  or  $g$  is a cofibration, we can change the other to be a cofibration, if necessary, without changing the homotopy type of any of the four spaces. Hence each map is an injection, and for a local coefficient system  $L$  on  $Y'$ , we have two long exact sequences in homology

$$\begin{array}{ccccccc} \longrightarrow & H_q(X, f^*g'^*L) & \xrightarrow{f_*} & H_q(Y, f'^*L) & \longrightarrow & H_q(Y, X; f'^*L) & \longrightarrow \dots \\ & \downarrow g_* & & \downarrow g'_* & & \downarrow (g, g')_* & \\ \longrightarrow & H_q(X', g'^*L) & \xrightarrow{f'_*} & H_q(Y', L) & \longrightarrow & H_q(Y', X'; L) & \longrightarrow \dots \end{array}$$

By hypothesis (1.2) (b) the morphism  $f_*$  is an isomorphism and thus  $H_*(Y, X; f'^*L) = 0$ . By excision  $(g, g')_*$  is an isomorphism and thus  $H_*(Y', X'; L) = 0$ . Hence  $f'_*$  is an isomorphism and criterion (1.2) (b) is satisfied for  $f'$  to be an acyclic map which proves the proposition.

The previous proposition concerning acyclic maps in a cofibration will be the basic tool for most of the results which follow in sections 2 and 3. It was pointed out to us by Quillen.

(2.4) PROPOSITION. *Consider the following diagram of CW-spaces.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow g' \\ X' & \xrightarrow{f'} & Y' \end{array}$$

*If  $g$  and  $g'$  are acyclic, and if  $\pi_1(f)$  and  $\pi_1(f')$  are isomorphisms then the diagram is cocartesian up to homotopy equivalence.*

*Proof.* First replace  $f$  and  $g$  by equivalent cofibrations and form  $h : X' \cup_X Y \rightarrow Y'$ . The map  $g'' : Y \rightarrow X' \cup_X Y$  is an acyclic map by (2.3) and  $g' = hg''$ . Thus  $h$  is acyclic by (2.1).

Since  $\pi_1(f)$  is an isomorphism, it follows that  $f'' : X' \rightarrow X' \cup_X Y$  has the property that  $\pi_1(f'')$  is an isomorphism by the van Kampen theorem and  $f' = hf''$ . Thus  $\pi_1(h)$  is an isomorphism. Now apply (1.5) to see that  $h$  is a homotopy equivalence. This proves the proposition.

(2.5) THEOREM. *Let  $f : X \rightarrow Y$  be an acyclic map between CW-spaces with homotopy fibre  $g : F \rightarrow X$ . Then  $f$  is the homotopy cofibre of  $g$ .*

*Proof.* Let  $CF$  be the cone over  $F$ . The homotopy cofibre  $C$  of  $g : F \rightarrow X$  is homotopy equivalent to  $CF \cup_F X$  and we have the cocartesian square

$$\begin{array}{ccccc} F & \xrightarrow{g} & X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow v & \nearrow h & \\ CF & \longrightarrow & C & & \end{array}$$

Since  $fg \simeq *$ , it follows that we have a map  $h : C \rightarrow Y$  such that  $f \simeq hv$ . Since  $f$  is acyclic, the map  $F \rightarrow CF$  is acyclic and, by (2.3)  $v$  is acyclic. One deduces then, by (2.1) that  $h$  is acyclic. As  $\pi_1(h)$  is onto (1.3), one has:

$$\ker(\pi_1(h)) = v(\ker \pi_1(f)) = v(\text{Im } \pi_1(g)) = 1$$

So  $\pi_1(h)$  is injective and, by (1.3) and (1.5),  $h$  is a homotopy equivalence.

(2.6) THEOREM. *Let  $f: X \rightarrow Y$  be an acyclic map between CW-spaces and let  $h_1, h_2: Y \rightarrow Z$  be two maps. If  $h_1 f \simeq h_2 f$ , then it follows that  $h_1 \simeq h_2$ .*

*Proof.* By (2.5) we have cofibre sequence

$$F \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \Delta F$$

where  $\Delta F$  is the reduced suspension of the acyclic space  $F$ . Since  $\Delta F$  is simply connected and  $\tilde{H}_*(\Delta F) = 0$ , it is contractible, and the group  $[\Delta F, Z]$  in the Puppe sequence is zero.

In general, the group  $[\Delta F, Z]$  acts transitively on the fibres of the function  $[Y, Z] \rightarrow [X, Z]$ , so that in this case,  $[Y, Z] \rightarrow [X, Z]$  is injective. This proves the theorem.

### § 3. CLASSIFICATION OF ACYCLIC MAP FROM A GIVEN SPACE

Let  $X$  be a path connected space. To each acyclic map  $f: X \rightarrow Y$ , we assign the kernel of  $\pi_1(f): \pi_1(X) \rightarrow \pi_1(Y)$  which is a perfect normal subgroup of  $\pi_1(X)$  by (1.3). The object of this section is to show that this map from isomorphism classes of acyclic maps defined on  $X$  to perfect normal subgroups of  $\pi_1(X)$  is a bijection.

(3.1) PROPOSITION. *Let  $f: X \rightarrow Y$  and  $f': X \rightarrow Y'$  be two maps between CW-spaces such that  $f$  is acyclic. There exists a map  $h: Y \rightarrow Y'$  with  $hf \simeq f'$  if and only if  $\ker \pi_1(f) \subset \ker \pi_1(f')$ , and such an  $h$  is unique up to homotopy. In addition, if  $f'$  is acyclic, then  $h$  is acyclic, and  $h$  is a homotopy equivalence if and only if  $\ker \pi_1(f) = \ker \pi_1(f')$ .*

*Proof.* If  $h$  exists, then  $\pi_1(f') = \pi_1(h) \circ \pi_1(f)$  and we have  $\ker \pi_1(f) \subset \ker \pi_1(f')$ . Conversely, we can suppose  $f$  is a cofibration and form the cocartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f' \downarrow & & \downarrow g' \\ Y' & \xrightarrow{g} & Y' \cup_X Y \end{array}$$