HOW QUICKLY CAN AN ENTIRE FUNCTION TEND TO ZERO ALONG A CURVE ?

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by W. K. Hayman

1. Introduction

Suppose that $f(z)$ is an entire function and that

$$M(r, f) = \sup_{|z|=r} |f(z)|$$

is the maximum modulus of $f(z)$. In this talk I should like to discuss how small $f(z)$ can become compared with $1/M(r)$ on a suitably large set E. Evidently $f(z) = 0$ at all the zeros of $f(z)$, so that we must not take E too small if we are to get a non-trivial result.

A classical problem concerns the minimum modulus

$$\mu(r, f) = \inf_{|z|=r} |f(z)|.$$

In our terminology this corresponds to a set E which meets every circle $|z| = r$. The quantity $\mu(r)$ was found to be important by Hadamard [2] in discussing the product representation for $f(z)$. We define the order λ or lower order μ of $f(z)$ by

$$\lambda = \overline{\lim_{r \to \infty}} \frac{\log \log M(r)}{\log r}, \quad \mu = \underline{\lim_{r \to \infty}} \frac{\log \log M(r)}{\log r}.$$

Let us suppose that $0 \leqslant \lambda < \infty$, let q be the integral part of λ and write

$$E(z, q) = (1-z) \exp\left\{z + \frac{1}{2} z^2 + \dots + \frac{1}{q} z^q\right\}.$$

If $f(z)$ has a zero of order p at the origin and other zeros z_ν each counted with correct multiplicity, we write

$$\Pi(z) = z^p \prod_{\nu=1}^{\infty} E\left(\frac{z}{z_\nu}, q\right).$$

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Then Hadamard showed that $\Pi(z)$ has order at most λ and

$$\mu(r, \Pi) > \exp(-r^{\lambda+\varepsilon})$$

for "most" values of r. Thus for such r

$$|F(z)| = \left|\frac{f(z)}{\Pi(z)}\right| < \exp(r^{\lambda+\varepsilon}), \quad |z| = r$$

and $F(z)$ is an entire function without zeros. Thus

$$F(z) = e^{P(z)}$$

where

$$Re\{P(z)\} < r^{\lambda+\varepsilon}, \quad |z| = r$$

and from this it is not difficult to show that $P(z)$ is a polynomial of degree at most λ. This yields the Hadamard product decomposition

(1) $$f(z) = e^{P(z)}\,\Pi(z).$$

The representation (1) is particularly useful when $\lambda < 1$, i.e. $q = 0$, when we obtain

(2) $$f(z) = az^p \prod_{\nu=1}^{\infty}\left(1 - \frac{z}{z_\nu}\right),$$

so that $f(z)$ has infinitely many zeros, unless f is a polynomial. It is also easy to deduce from (2) that the order λ depends only on the moduli of the zeros. Thus if we write $r_\nu = |z_\nu|$,

(3) $$F(z) = |a|\, z^p \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{r_\nu}\right)$$

we deduce that

(4) $$|F(-r)| \leqslant \mu(r, f) \leqslant M(r, f) \leqslant F(r)$$

and

$$|F(r)F(-r)| \leqslant \mu(r, f)\, M(r, f).$$

These inequalities enable us to reduce the problem of the behaviour of functions of order less than one in most cases to that of the functions (3) which have all their zeros on the negative axis. Thus Valiron [9] and Wiman [10] proved the sharp result

(5) $$\mu(r) > M(r)^{\cos(\pi\lambda)-\varepsilon}$$

for a sequence of r tending to ∞. This had been conjectured by Littlewood [8] who proved the corresponding theorem with $\cos(2\pi\lambda)$ instead of $\cos(\pi\lambda)$. The result is valid for $0 \leqslant \lambda \leqslant 1$.

If $1 < \lambda < \infty$, Littlewood [8] also proved that there exists a positive constant $C(\lambda)$ such that

$$\mu(r) > M(r)^{-C(\lambda)-\varepsilon}$$

for a sequence of r tending to ∞. However the correct value of $C(\lambda)$ is unknown for $\lambda > 1$. It turns out that the formula (1) with exponential factors is much harder to work with than (2). Wiman [11] conjectured that $C(\lambda) = 1$ for $\lambda > 1$, a result which is true if $f(z)$ has no zeros. Later Beurling [1] proved a corresponding theorem for the case when $f(z)$ attains its minimum on a ray. Nevertheless Wiman's conjecture is false and the correct order of magnitude of Littlewood's constant $C(\lambda)$ is $\log \lambda$ as $\lambda \to \infty$. For infinite order the corresponding Theorem is [4].

$$\mu(r) > M(r)^{-A \log \log \log M(r)}, \tag{6}$$

where the best value of A lies between .09 and 11.03.

Since the theory of $\mu(r)$ is thus rather unsatisfactory for $\lambda > 1$ it is natural to consider other cases of E. Suppose first that E is a ray $\arg z = \theta$ and that $K > 1$. Then Beurling [1] showed that if

$$|f(re^{i\theta})| < M(r)^{-K}, \tag{7}$$

for $0 < r < R$, we have

$$|f(z)| < 1, \quad |z| = C_1(K)R,$$

where the constant $C_1(K)$ depends only on K. If R can be chosen arbitrarily large, we deduce at once that $f(z)$ is bounded on a sequence of large circles $|z| = C_1 R$, so that f is constant by Liouville's theorem. Thus for non-constant f (7) cannot be true for all r (or all large r) and a fixed θ.

2. The case when E is a curve

It is natural to consider the case when E is an unbounded connected set or equivalently a curve going to ∞ and this is the topic I mainly wish to discuss today. By a rather involved method I had shown [4] that in this case

$$|f(z)| > M(r)^{-A_0}, \tag{8}$$

for some arbitrarily large $z = re^{i\theta}$ on E. Here A_0 is an absolute but presumably very large constant. I had conjectured that the result holds for any $A_0 > 1$. Soon afterwards Beurling showed Kjellberg in a conversation that (8) holds for any $A_0 > 3$. Beurling's argument is as follows.
We write

$$B(r) = \log^+ M(r) = \max\{0, \log M(r)\}, \quad B(z) = B(|z|),$$

and suppose that for some $K \geqslant 1$, we have

$$\log |f(z)| < -KB(z), \tag{9}$$

on a Jordan curve Γ joining $z = 0$, $z_0 = Re^{i\theta}$. Then we deduce that

$$\log |f(re^{i\theta})| \leqslant -\frac{K-1}{2} B(r), \quad 0 < r < R. \tag{10}$$

To see this we suppose that S: $[r_1, r_2]$ is a maximal interval such that $re^{i\theta}$ does not lie on Γ, for $r_1 < r < r_2$. Let γ be the arc of Γ with end points $r_1 e^{i\theta}$, $r_2 e^{i\theta}$, let D be the domain bounded by γ and S, D^* the reflexion of D in S and $\Delta = D \cup S \cup D^*$. In Δ we consider the function

$$u(z) = \log |f(z)| + \log |f(z^*)| + (K-1)B(z)$$

where z^* is the reflexion of z in S. Clearly $u(z)$ is subharmonic in Δ and, for z on the boundary of Δ, either z or z^* lies on Γ. Thus

$$u(z) \leqslant 0$$

in Δ and in particular on S. We deduce that

$$2 \log |f(re^{i\theta})| \leqslant -(K-1)B(r), \quad r_1 < r < r_2$$

and this yields (10). Hence if $K > 3$, we deduce that f is constant from Beurling's theorem.

Recalling his earlier conversation with Beurling, Kjellberg went on to prove 18 months ago that (8) holds for any $A_0 > 1$ at least when f has finite order and I managed to extend the result to the case of infinite order. Our joint paper will be published in the Turan memorial volume. I should like to describe briefly the idea behind this proof.

3. An extended reflexion principle

Let us return to the above reflexion argument. We assume now that (9) holds on some curve Γ going from 0 to ∞, where $K \geqslant 1$. Then the reflexion principle shows that

$$\log |f(z)| \leqslant -\frac{K-1}{2} B(z) \tag{11}$$

on any ray joining the origin to some point on Γ. Kjellberg extended this to prove the following

Lemma. *If f has lower order $\mu < \infty$. Then (11) holds in some sector of opening at least π/μ.*

From this he was able to obtain a contradiction if $K > 1$. To prove the Lemma we let θ_1, θ_2 be the lower and upper limits of $\arg z$ as $z \to \infty$ on Γ. Then the above argument shows that (11) holds for $\theta_1 < \arg z < \theta_2$. Thus is $\theta_2 - \theta_1 \geqslant \pi/\mu$, the Lemma is proved.

Suppose now that $\theta_2 - \theta_1 < \pi/\mu$. We may assume that $\mu \geqslant 1$, since otherwise our conclusion follows from (5) in which λ can be replaced by μ according to a Theorem of Kjellberg [7]. We choose a sequence R_n which tends to ∞ with n and is such that

$$\log B(R_n) < (\mu + o(1)) \log R_n . \tag{12}$$

We now define quantities α_1, α_2 as follows. For any fixed $\phi_1 < \theta_1$ and sufficiently large R, we define $h_1(R, \phi_1)$ to be the largest number such that the arc

$$\phi_1 < \arg z < \phi_1 + h_1(R, \phi_1), \quad |z| = R$$

does not meet Γ. Clearly $h_1 \leqslant \theta_2 + o(1) - \phi_1$ for large R. Similarly, for $\phi_2 > \theta_2$, we define $h_2(R, \phi_2)$ to be the largest number such that the arc

$$\phi_2 - h_2(R, \phi_2) < \arg z < \phi_2, \quad |z| = R$$

does not meet Γ. Then α is defined to be the greatest lower bound of all $\phi_1 < \theta_1$ such that, for a fixed large R_0, we have

$$\varliminf_{n\to\infty} \frac{1}{\log R_n} \int_{R_0}^{R_n} h_1(t, \phi_1) \frac{dt}{t} < \frac{\pi}{2\mu} .$$

If there are no such numbers ϕ_1, we define $\alpha_1 = \theta_1$. Also α_2 is defined similarly as the least upper bound of all $\phi_2 > \theta_2$ such that

$$\varliminf_{n\to\infty} \frac{1}{\log R_n} \int_{R_0}^{R_n} h_2(t, \phi_2) \frac{dt}{t} < \frac{\pi}{2\mu} . \tag{13}$$

If there are no such ϕ_2 we define $\alpha_2 = \theta_2$.

Suppose now that $\phi_1 < \alpha_1$, $\phi_2 > \alpha_2$. Then we deduce that for a fixed large R_0 and all sufficiently large n

$$(\phi_2 - \phi_1) \log \frac{R_n}{R_0} = \int_{R_0}^{R_n} (\phi_2 - \phi_1) \frac{dt}{t} \geqslant \int_{R_0}^{R_n} \{h_1(t, \phi_1) + h_2(t, \phi_2)\} \frac{dt}{t}$$
$$\geqslant \left(\frac{\pi}{\mu} + o(1)\right) \log R_n .$$

Thus $\phi_2 - \phi_1 \geqslant \pi/\mu$, and hence $\alpha_2 - \alpha_1 \geqslant \pi/\mu$.
On the other hand we can show that (11) holds for $0 < |z| < \infty$, $\alpha_1 \leqslant \arg z \leqslant \alpha_2$.

To see this we choose ϕ, such that $\alpha_1 < \phi < \alpha_2$ and assume that Γ does not meet the ray $\arg z = \phi$ for arbitrarily large z, since otherwise the conclusion follows from (10). In particular (11) holds for $\theta_1 < \phi < \theta_2$ and hence by continuity also for $\phi = \theta_1$ or θ_2. Thus we may assume that either $\alpha_1 < \phi < \theta_1$ or $\theta_2 < \phi < \alpha_2$. Suppose e.g. that the latter inequality holds, so that in particular $\alpha_2 > \theta_2$. Let $z_0 = Re^{i\phi}$ be the last intersection of $\arg z = \phi$ with Γ. Let D be the domain bounded by the arc Γ_0 of Γ from z_0 to ∞ and by the segment $S: z = te^{i\phi}$, $R_0 < t < \infty$.
Let D^* be the reflexion of D in S and set $\Delta = D \cup S \cup D^*$.
We consider

$$u(z) = \log |f(z)| + \log |f(z^*)| + (K-1) B(z)$$

in Δ, where z^* denotes the reflexion of z in S, and proceed to show that

$$(14) \qquad u(z) \leqslant 0 \text{ in } \Delta .$$

By our construction (14) holds on the finite boundary $\Gamma_0 \cup \Gamma_0^*$ of Δ.
To deal with points at ∞ we combine (12) and (13).
We choose a large n and define $\omega_n(z)$ to be the harmonic measure of the circle $|z| = R_n$, with respect to the subdomain Δ_n of Δ bounded by $|z| = R_n$, Γ_0, Γ_0^* and containing the part $R_0 < t < R_n$ of the segment S.
If z is a fixed point of and we let n tend to ∞, then standard estimates yield [1])

$$(15) \quad \omega_n(z) \leqslant \exp\left\{-\pi \int_{R_0+1}^{R_n} \frac{dt}{2 h_2(t, \phi)} + O(1)\right\}, \quad \text{as } n \to \infty .$$

[1]) We may map Δ_n onto a half strip and then apply Ahlfors' distortion theorem in the form given in [3].

Also Schwarz's inequality yields

$$\int_{R_0+1}^{R_n} h_2(t,\phi)\,\frac{d\,t}{t}\int_{R_0+1}^{R_n}\frac{d\,t}{t\,h_2(t,\phi)} \geqslant \left\{\log\left(\frac{R_n}{R_0+1}\right)\right\}^2,$$

i.e.

$$\int_{R_0+1}^{R_n}\frac{d\,t}{t\,h_2(t,\phi)} \geqslant \left\{\log\frac{R_n}{R_0+1}\right\}^2 \Big/ \int_{R_0+1}^{R_n} h_2(t,\phi)\,\frac{d\,t}{t}$$

$$> \frac{2\,\mu+2\,\delta}{\pi}\log\, R_n$$

for all large n, where δ is a positive constant, in view of (13). Thus (15) yields

$$\omega_n(z) = O(R_n{}^{-\mu-\delta}), \quad \text{as } n \to \infty\,. \tag{16}$$

Also since $u(z) \leqslant (K+1)\,B(R_n)$ on $|z| = R_n$, we deduce finally that

$$u(z) \leqslant (K+1)\,B(R_n)\,\omega_n(z)$$

in Δ_n and now (12) and (16) yield (14) for any point in Δ. In particular for z on S, we deduce (11) as required. This proves the Lemma.

4. Conclusions

It is not difficult to obtain a contradiction from the above Lemma. We may assume without loss of generality that the angle is given by $S: |\arg z| < \dfrac{\pi}{2\,\mu}$. Since $f(z)$ is bounded in S, we deduce that $\log|f(z)|$ is bounded above in S by the Poisson integral of the boundary values on the arms $\arg z = \mp\, \pi/(2\mu)$. This leads, for $K > 1$, to

$$\log\,|f(re^{i\theta})| < -\,A(\mu)(K-1)\,r^{\mu}\int_r^{\infty}\frac{B(t)\,d\,t}{t^{\mu+1}}, \quad |\theta| < \frac{\pi}{2\,\mu}, \tag{17}$$

$$0 < r < \infty\,,$$

where the constant $A(\mu)$ depends only on μ.

Given any constant $C > 1$, we can, since f has lower order μ find a sequence r_n tending to infinity with n and such that

$$B(t) > \frac{1}{2}\left(\frac{t}{r_n}\right)^{\mu} B(r_n)\,, \quad r_n \leqslant t \leqslant Cr_n$$

Now (17) yields

$$\log |f(r_n e^{i\theta})| < - A(\mu)(K-1)B(r_n)\frac{1}{2}\int_{r_n}^{Cr_n}\frac{dt}{t}$$
$$= -\frac{1}{2}A(\mu)(K-1)B(r_n)\log C .$$

Thus

$$\int_{-\pi}^{\pi}\log |f(r_n e^{i\theta})|\,d\theta \leqslant -\frac{\pi}{\mu}A(\mu)(K-1)B(r_n)\log C$$
$$+\left(2\pi-\frac{\pi}{\mu}\right)B(r_n) .$$

This contradicts Jensen's formula if C is sufficiently large, since the left hand side is bounded below.

We can also obtain some conclusions if $K = 1$. In this case we note that if $S: \alpha_1 \leqslant \arg z \leqslant \alpha_2$ is the angle constructed in Lemma 1 then, since Γ lies almost entirely in S, we deduce for large r that

$$\inf_{\alpha_1 \leqq \theta \leqq \alpha_2} \log |f(re^{i\theta})| \leqslant -\frac{1}{3}B(r) .$$

Since f is bounded above in S it follows from an earlier Theorem of mine [5] that

$$\overline{\lim_{r\to\infty}}\; B(r)\, r^{\pi/(\alpha_2-\alpha_1)} < \infty .$$

Thus $B(r)$ has order $\lambda = \mu$ and f cannot have maximal type. Further $\alpha_2 - \alpha_1 = \pi/\lambda$ and from this we deduce that as $z = re^{i\theta} \to \infty$ on Γ outside a set of r of logarithmic density zero

$$\theta = arg\ z \to \frac{1}{2}(\alpha_1+\alpha_2) ,$$

so that Γ has a preferred direction. If $\mu = \infty$, we must have $\alpha_1 = \alpha_2$, so that Γ has a unique limiting direction.

We also note that $\mu > 1$, unless $f(z) \equiv e^{(az+b)}$. For we have seen that f cannot have order 1, maximal type. However if $f(z) \not\equiv e^{az+b}$ and f has order one mean type, or minimal type then an earlier theorem of mine [6] shows that $\mu(r)\, M(r)$ cannot be bounded.

Finally let me say a few words concerning the case of infinite order. In this case we assume $K > 1$ and define

$$\mu(r) = \inf \frac{\log B(r_2) - \log B(r_1) + A_1(K)}{\log r_2 - \log r_1}$$

where the inf is taken over all pairs r_1, r_2, such that $r < r_1 < r_2 < \infty$, and

$$A_1(K) = \log\left\{\frac{20(1+K)}{K-1}\right\}.$$

The quantity $\mu(r)$ plays a similar role to the lower order μ in the above argument and

$$\log |f(z)| < -\frac{K-1}{4} B(z), \ \alpha_1(r) < arg\ z < \alpha_2(r), \ |z| < r,$$

where $\alpha_2 - \alpha_1 \geqslant \pi/\mu(r)$. From this and the fact that $\mu(r)$ increases with r it is possible to obtain a contradiction.

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