2. General constructions on D and E-Modules

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M.1) p_j is homogeneous of degree j in ξ.

M.2) $\sup |p_j(x, \xi)| \leqslant (-j)!\ R_K^{-j}$ for any $K \subset\subset U$, and $j < 0$.

M.3) $p_j = 0$ for $j >> 0$.

At a point $(x, 0)$, the p_j are homogeneous polynomials of degree j in ξ; therefore, $p_j = 0$ for $j < 0$, and, for $j \geqslant 0$, p_j can be identified with the differential operator $p_j(x_i, \partial_i)$; the formulae for multiplication and change of variables in $\mathscr{E}$ are chosen in order to extend what happens on $\mathscr{D}$. In that way, one get a sheaf $\mathscr{E}$ on T^*X with a filtration $\mathscr{E}_j, j \in \mathbf{Z}$ and a structure of (flat) $\pi^*(\mathscr{D})$-Module. All the properties of $\mathscr{D}$ mentioned before can be extended to $\mathscr{E}$, which is called the sheaf of (convergent) microdifferential operators. Note also the following property: if $p \in \mathscr{E}(U)$ has a symbol $\sigma(p)$ which does not vanish, then p is invertible in $\mathscr{E}(U)$ [3]; from that results easily the following useful property: if M is a coherent $\mathscr{D}$-Module, one has char M = support of $\tilde{M}$ with $\tilde{M} = \mathscr{E} \otimes_{\pi^{-1}\mathscr{D}} \pi^{-1} M$.

A variant of the preceding sheaf with essentially similar properties, is given by the sheaf $\hat{\mathscr{E}}$ of "formal" microdifferential operators (it is defined like $\mathscr{E}$, by just removing M.2). Perhaps this sheaf, or an algebraic counterpart, could have some interest for an *algebraic* theory of $\mathscr{D}$-Modules.

2. General constructions on $\mathscr{D}$ and $\mathscr{E}$-Modules

(2.1) *Canonical transformations.*

This operation is restricted to $\mathscr{E}$-Modules on open sets $U \subset T^*X - X$; this is the analytic counterpart of Maslov's ideas [13] and of the theory of "Fourier integral operators" by Hörmander [7]. Given a homogeneous symplectic diffeomorphism $U \xrightarrow{\varphi} V$, with $U, V \subset T^*X - X$, there exists a (non-unique) isomorphism $\mathscr{E} | U \rightarrow \mathscr{E} | V$, which respects the filtrations, and verifies $\sigma\Phi(P) = \sigma(P) \circ \varphi^{-1}$. This is often useful to reduce the support of an $\mathscr{E}$-Module, at least at smooth points, to canonical form. Although this is a very fundamental ingredient of the theory, we will not insist on it here. We just mention that Φ is defined by a suitable holonomic system, whose support (= characteristic variety) is precisely the graph of φ in $U \times V$. For the details, we refer to [S.K.K.]; see also [B.L.M.].

(2.2) *Direct images.*

We introduce first some definitions; let Y be another manifold of dimension p, and $f: X \to Y$ a holomorphic mapping; we define $\mathscr{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathscr{D}_Y)$; this sheaf on X is nothing but the sheaf of differential operators from $f^{-1}(\mathcal{O}_Y)$ into $\mathcal{O}_X$; therefore it has a structure of left $\mathscr{D}_X$-Module and right $f^{-1}(\mathscr{D}_Y)$-Module; we leave to the reader the explicit definition of these structures. Similarly note that Ω^n_X, the sheaf of holomorphic n forms on X is a right $\mathscr{D}$-Module (by the following action, if ξ is a vector field, and $\alpha \in \Omega^n$, we write $\alpha\xi = -\theta_\xi \alpha$, θ the Lie derivative); we define therefore the sheaf on X of $(f^{-1}(\Omega^p_Y), \Omega^n_X)$-differential operators by $\mathscr{D}_{Y\leftarrow X} = f^{-1}(\mathscr{D}_Y \otimes_{\mathcal{O}_Y} (\Omega^p_Y)^{-1}) \otimes_{f^{-1}(\mathcal{O}_Y)} \Omega^n_X$ [here we use the right structure of $\mathscr{D}_Y$-Module over $\mathcal{O}_Y = \mathscr{D}_{0Y}$]; it has a structure of right $\mathscr{D}_X$-Module and left $f^{-1}(\mathscr{D}_Y)$-Module.

Now, let M be a left coherent $\mathscr{D}_X$-Module; the direct images (or "integration" in the fiber) are defined by $\int^i M = R^i f_* (\mathscr{D}_{Y\to X} \overset{L}{\otimes}_{\mathscr{D}_X} M)$, where R (resp. L) denotes the right (resp. left) derived functors. To understand the meaning of these operations, we will examine special cases.

i) *Case where Y is a point* (the "absolute" case).

Here, one has $\mathscr{D}_{Y\leftarrow X} = \Omega^n_X$; on the other hand, denote by $DR^\cdot(M)$ the "de Rham complex of M"

$0 \to M \to M \otimes_{\mathcal{O}_X} \Omega^1_X \overset{d}{\to} \dots \overset{d}{\to} M \otimes_{\mathcal{O}_X} \Omega^n_X \to 0$, where d is the usual exterior derivative; it is easy to verify that one has an isomorphism $\Omega^n_X \otimes^L_{\mathscr{D}_X} M \overset{\sim}{\to} DR^\cdot(M)[n]$ (where $[n]$ means "shifted n times to the left"); and, one has also an isomorphism $DR^\cdot(M) \simeq \underline{\underline{R}}\,\mathrm{Hom}_{\mathscr{D}_X}(\mathcal{O}_X, M)$; therefore, one has

$$\int^i M = \underline{\underline{H}}^{i+n}(X, DR^\cdot(M)) = \mathrm{Ext}^{i+n}_{\mathscr{D}_X}(X; \mathcal{O}_X, M).$$

Therefore, here, the direct image is the global de Rham hypercohomology of M, with a shifting of the degree by n.

ii) *The case where $X \to Y$ is smooth* (i.e. is locally a submersion).

This case is similar: one gets the relative de Rham cohomology (with a shifting by $n-p$). Note that we have automatically a structure of left $\mathscr{D}_Y$-Module on $\int^i M$; in the case where $M = \mathcal{O}_X$, this structure is just the so-called "Gauss-Manin connection".

iii) *The case where X is a submanifold of Y.*

In local coordinates, we can suppose $p = n + k$, $x_i = y_i$, $1 \leqslant i \leqslant n$, and that X is defined by $y_i = 0$, $i \geqslant n+1$. Then one has

$\mathscr{D}_{Y \leftarrow X} \simeq \mathscr{D}_Y / \sum_{i \geq n} \mathscr{D}_Y y_i$; elements of $\mathscr{D}_{Y \leftarrow X}$ can be written as $\Sigma\, \mathscr{D}^{\alpha}_{y'} \,.\, P_\alpha$, with $y' = (y_{n+1}, \ldots, y_p)$, $P_\alpha \in \mathscr{D}_X$; this is a free Module over $\mathscr{D}_X$, and f_* here is exact; therefore $\int^i M = 0$, $i \neq 0$, and $\int^0 M$ is just the $\mathscr{D}_Y$-Module whose elements can be written uniquely as $\Sigma\, D^{\alpha}_{y'} \otimes m_{\alpha'}$ $m_\alpha \in \mathscr{D}_X$. This is just the same correspondence as in the theory of distributions: "distributions on X" $\rightarrow$ "distributions on Y with support in X".

Now, one can prove that formation of $\int^{\cdot}$ is compatible with composition (i.e. one has an isomorphism of derived functors $\int^{\cdot}_{fg} \simeq \int^{\cdot}_{f} \cdot \int^{\cdot}_{g}$); then, the general case reduces to ii) and iii).

The following theorem is due to Kashiwara [10].

THEOREM 2.2.1. *Suppose* f *projective* (i.e. *proper and factorizing through some closed embedding* $X \rightarrow Y \times \mathbf{P}_k(\mathbf{C})$, *and suppose that* M *has a global good filtration. Then*

i) *The* $\int^i M$ *are coherent* $\mathscr{D}_Y$-*Modules.*

ii) *The characteristic variety of* $\int^i M$ *is contained in the set of* $\eta \in T^* Y$, *with* $y = \pi(\eta)$, *such that there exists* $\xi \in$ char M, *with* $x = \pi(\xi) \in X$ *verifying* $y = f(x)$, $\xi = Tf^*_x(\eta)$; *here* Tf^* *denotes the cotangent map of* f.

If M is holonomic, and the other hypotheses of the theorem are satisfied, this implies easily that the $\int^i M$ are holonomic.

The proof of ii) requires some microlocalization of the notion of direct images, which I will not develop here. Also, it is likely that the hypothesis "f proper" is sufficient for the conclusions of the theorem. Perhaps, it is also true that one has coherence of local direct images of *holonomic* Modules, when one replaces X by a small ball, as in Milnor's work on singularity of hypersurfaces, and in the study of local Gauss-Manin connection by several authors (Brieskorn [4], Hamm [6], etc.); this is at least true in the absolute case (see § 3).

(2.3) *Inverse images, and localization.*

Let $f: X \rightarrow Y$, as before, and M a coherent left $\mathscr{D}_Y$-Module; as in analytic geometry, one defines $f^* M = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(M)$; the obvious isomorphism:

$f^* M \simeq \mathscr{D}_{X \to Y} \otimes_{f^{-1}(\mathscr{D}_Y)} f^{-1}(M)$ provides $f^* M$ with a structure of $\mathscr{D}_X$-Module; the left derived functors $L_i f^* M$ are defined in the same way, with "Tor".

Again, the study is reduced to two cases: i) submersions, ii) closed embeddings; the first case is trivial, therefore we consider only the second and suppose, from now on, that X is a closed submanifold of Y. In that case, the $L_i f^* M$ are not coherent in general (take for instance $M = \mathscr{D}_Y$, and X defined by $y_p = 0$). There are three cases of interest:

a) *The non-characteristic case.*

One says that X is non-characteristic with respect to M if char $M \cap N^*X$ is contained in the zero-section (N^* denotes the conormal bundle). This is a well-known notion, f.i. in connection with the Cauchy-Kovalevskaya theorem. Then, if X is non-characteristic, $f^* M$ is coherent, and $L_i f^* M = 0$, $i \geqslant 1$. Moreover, one has char $f^* M = (Tf)^*$ (char M). See [S.K.K.].

b) *The case where M has support in X.*

In that case, one has $L_i f^* M = 0$, $i \neq d = p - n$ and $L_d f^* M$ is a coherent $\mathscr{D}_X$-Module; we will denote it by $\bar{f}^* M$; in local coordinates, $x_i = y_i$, $1 \leqslant i \leqslant n$, X defined by $y_{n+1} = \ldots = y_p = 0$, $\bar{f}^* M$ is the set $\overline{M}$ of $m \in M$ annihilated by $y_{n+1}, \ldots, y_p$ (take the resolution of $\mathscr{O}_X$ over $\mathscr{O}_Y$ by the Koszul complex), but this is not intrinsic; $\overline{M}$ has no canonical structure of $\mathscr{D}_X$-Module, and has to be tensorised by a suitable invertible sheaf on $\mathscr{O}_X$ to become $\bar{f}^* M$.

One remarkable phenomenon occurs: M is canonically isomorphic with $\int^0 \bar{f}^* M$; in other words, the functors $M \mapsto \bar{f}^* M$ and $N \mapsto \int^0 N$ give an equivalence between the category of coherent $\mathscr{D}_Y$-Modules with support in X and the category of $\mathscr{D}_X$-Modules, a situation much simpler than in usual analytic geometry. For instance, in local coordinates, the coherent $\mathscr{D}_X$-Modules with support o are finite sums of copies of $\mathscr{D}_X \delta \simeq \mathscr{D}_X / \Sigma \mathscr{D}_X x_i$ (this module is also well-known to algebraists as the injective envelope of $\mathbf{C}$ over $\mathbf{C}\{x_1, \ldots, x_n\}$).

One has $\dim N - n = \dim \int^0 N - p$; in particular, holonomy is preserved in this correspondence. For these results, see [8] or [B.L.M.].

c) *The case where M is holonomic.*

In this case, one has the following result, much more difficult than the preceding ones:

THEOREM (2.3.1). *If* M *is holonomic on* Y, *then the* $L_i f^* M$ *are holonomic on* X (Kashiwara [11]).

However, one problem here is to find the characteristic varieties (this restriction seems to have no microlocal counterpart). Note also that, in the case of modules over the Weyl algebra, i.e. the algebra of differential operators on $\mathbf{C}^n$ with *polynomial* coefficients, the holonomy of $f^* M$ was proved previously by I.N. Bernstein [1].

The preceding theorem can be stated in a more general context, using local cohomology. If now Z is a closed analytic subset of Y, defined by a coherent $\mathcal{O}_Y$-Ideal J, we define $H^i_{[Z]} M = \varinjlim \operatorname{Ext}^i_{\mathcal{O}_X} (\mathcal{O}_X/J^k, M)$; this is not the "transcendental" local cohomology $H^i_Z M$, but the analytical translation of the local cohomology of schemes; it is easily provided with a structure of $\mathcal{D}_Y$-Module. Now, if $X \subset Y$ is a submanifold, it is easy to prove that one has $L_i f^* M = \bar{f}^* (\underline{H}^{d-i}_{[X]} M)$, with $d = \operatorname{codim}_Y X$. Therefore, theorem (2.3.1) is a special case of the following theorem (same reference):

THEOREM (2.3.3). *If* M *is holonomic, then the* $H^i_{[Z]} M$ *are holonomic.*

As an easy consequence, the sheaf of meromorphic sections of a connection with singularities in the sense of Deligne [5] is a holonomic $\mathcal{D}$-Module. In some sense, they are the "general case" of holonomic $\mathcal{D}$-Modules (a problem is to give a meaning to this assertion). In particular, modulo non-singular compactifications, one deduces immediately from that fact the following theorem, proved previously by Björk (unpublished ?): the algebraic de Rham cohomology of an algebraic connection on an affine non-singular $\mathbf{C}$-variety is finite.

3. FURTHER RESULTS ON HOLONOMIC SYSTEMS

First, note that, if M is a coherent left $\mathcal{D}$-Module on X and N any $\mathcal{D}$-Module, then $\operatorname{Hom}_{\mathcal{D}} (M, N)$ can be interpreted as the set of solutions of the system of p.d.e. defined by M, with values in N (for instance, if J is a left coherent sheaf of ideals of $\mathcal{D}$, and $M = \mathcal{D}/J$, then $\operatorname{Hom}_{\mathcal{D}} (M, N)$ is the set of $n \in N$ annihilated by J). For instance, taking $N = \mathcal{O}$, we get the holomorphic solutions of M; on the other hand, we have seen the relation between $\mathcal{R} \operatorname{Hom} (\mathcal{O}, N)$ and the de Rham cohomology of N.