# 3. The formal vector fields and the diagonal complex


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through a map $H^*(BG; R) \to H^*(L_M; G)$ so that we get a commutative diagram

$$\begin{array}{ccc} & & H^*(L_M; G) \\ & \nearrow & \\ H^*(BG; R) & & \downarrow \\ & \searrow & \\ & & H^*(X; R) \end{array}$$

So it is important to compute the map $H^*(BG; R) \to H^*(L_M; G)$. When $G$ is a compact connected Lie group, then $H^*(BG; R)$ is the algebra $I(G)$ of invariant polynomials on the Lie algebra of $G$, and the map from $I(G)$ to $C^*(L_M; G)$ is given by a $G$-connexion in $C^*(L_M)$ (cf. [5]).

In the example above, namely $M = S^1$ and $G = SO_2$, then $H^*(BSO_2)$ is a polynomial algebra in a generator of degree 2, the Euler class, which is mapped on a non zero multiple of $e$.

## 3. The formal vector fields and the diagonal complex

Given a point $x$ on $M$, we can consider the Lie algebra $L_M^x$ of infinite jets at $x$ of vector fields on $M$ with the quotient topology. It is isomorphic to the Lie algebra $\mathfrak{a}_n$ of formal vector fields $\sum v_i(x)\, \partial/\partial x^i$ in $R^n$, where the $v_i(x)$ are formal power series in the coordinates $x^1, \ldots, x^n$.

The natural map $L_M \to L_M^x$ associating to a vector field its jet at $x$ gives a $DG$-algebra morphism

$$C^*(L_M^x) \to C^*(L_M)$$

where $C^*(L_M^x)$ is the algebra of multilinear alternate forms on $L_M^x$ depending only on finite order jets.

The first and most important step in the work of Gelfand-Fuks was the complete determination of the cohomology $H^*(\mathfrak{a}_n)$ of the topological Lie algebra of formal vector fields on $R^n$.

Theorem 1. (Gelfand-Fuks [8], [9]). *Let* $E(h_1, \ldots, h_n)$ *be the exterior algebra on generators* $h_i$ *of degree* $2i-1$ *and let* $R[c_1, \ldots, c_n]\hat{}_{2n}$ *be the quotient of the polynomial algebra in generators* $c_i$ *of degree* $2i$ *by the ideal of elements of degree* $> 2n$.

*Then a model for* $C^*(\mathfrak{a}_n)$ *is the DG-algebra*

$$WU_n = E(h_1, ..., h_n) \otimes R[c_1, ..., c_n] \hat{}_{2n}$$

*with* $dh_i = c_i$ *and* $dc_i = 0$.

It follows that $H^i(\mathfrak{a}_n) = 0$ for $1 \leqslant i \leqslant 2n$ and $i > n^2 + 2n$. Also the multiplicative structure is trivial; more precisely, $WU_n$ is a model for a wedge of spheres (for instance $S^3$ for $n = 1$, $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ for $n = 2$) (cf. Vey [9]).

$WU_n$ is also a model for the space $F_n$ obtained by taking the restriction of the $U_n$-universal bundle over the $2n$-skeleton of its base space $BU_n$ (cf. Gelfand-Fuks [8]). Note that this representation is compatible with the natural actions of $O_n \subset U_n$.

One can also consider the relative complex $C^*(\mathfrak{a}_n, O_n)$ or $C^*(\mathfrak{a}_n, SO_n)$ of $O_n$ or $SO_n$-basic elements in $C^*(\mathfrak{a}_n)$, where $O_n$ is the orthogonal group acting in the usual way on $R^n$, hence on $\mathfrak{a}_n$.

Define $WO_n$ as the subalgebra of $WU_n$ generated by the $h_i$ with $i$ odd and all the $c_i$. From theorem 1, it is easy to deduce the

THEOREM 1′ [12]. *$WO_n$ is a model for $C^*(\mathfrak{a}_n, O_n)$.*
*A model for $C^*(\mathfrak{a}_n, SO_n)$ is $WO_n$ for $n$ odd and*

$$WSO_n = WO_n \otimes R[e] / (e^2 - c_n)$$

*for $n$ even, where* $\deg e = n$ *and* $de = 0$.

From the finite dimensionality of $H^*(\mathfrak{a}_n)$, using a suitable spectral sequence, Gelfand and Fuks prove in particular [7].

THEOREM 2. *If $H^*(M)$ is finite dimensional, then $H^*(L_M)$ is finite dimensional in each degree.*

*The Guillemin-Losik double complex.*

First define $C^*(L_M, \Omega_M)$ as the algebra of continuous alternate multilinear forms on $L_M$ with values in the algebra $\Omega_M$ of differential forms on $M$. We have two differentials, the first one defined as in 1 and the second one by the exterior differential in $\Omega_M$. So this is a double complex and we can consider the associated total differential.

$C^*_\Delta(L_M, \Omega_M)$ is the subcomplex of $C^*(L_M, \Omega_M)$ of those forms associating to a sequence $v_1, ..., v_k$ of vector fields on $M$ a differential form $f(v_1, ..., v_k)$ whose value at $x \in M$ depends only on finite order jets of the $v_i$'s at $x$.

THEOREM 3. (Guillemin [10], Losik [17]). *$C^*_\triangle(L_M, \Omega_M)$ is a model for a bundle $E$ with fiber $F_n$, base space $M$, associated to the tangent bundle of $M$.*

*More precisely, a model for $C^*_\triangle(L_M, \Omega_M)$ is the DG-algebra $\Omega_M \otimes WU_n$ over $\Omega_M$, where*

$$d(1 \otimes c_i) = 0 \quad d(1 \otimes h_i) = 1 \otimes c_i - p_{i/2} \otimes 1$$

*where $p_{i/2}$ is zero if $i$ is odd and is a form representing the Pontrjagin class of $M$ of degree $2i$ if $i$ is even.*

Note that if a foliation $F$ on $X \times M$ transverse to the fibers $\{x\} \times M$ is given, one has a characteristic homomorphism

$$C^*(L_M, \Omega_M) \to \Omega_{X \times M}$$

One has also a morphism

$$WO_n \to C^*_\triangle(L_M, \Omega_M)$$

(or $WU_n \to C^*(L_M, \Omega_M)$ in case $M$ has trivial Pontrjagin classes) whose composition with the previous one is the usual characteristic homomorphism for the foliation $F$ (cf. [3], [12]).

## 4. MAIN THEOREM

THEOREM 1. *$C^*(L_M)$ is a model for the space $\Gamma$ of continuous sections of the bundle $E$ described in the theorem above.*

This result, first conjectured by Bott (and also Fuks), has been proved by several people (Bott-Segal [1]), Fuks-Segal, Haefliger [13], Ph. Trauber, and others).

Suppose that $G$ is a compact connected Lie group acting on $M$. Then it also acts on the bundle $E$ and on its space of sections. Let us denote by $\Gamma_G$ the total space of the bundle with fiber $\Gamma$ associated to the universal $G$-bundle with base space $BG$.

THEOREM 1'. *$C^*(L_M; G)$ is a model for the space $\Gamma_G$.*

The way I proved theorem 1 was to construct first a tentative algebraic model $A$ for $\Gamma$ following ideas of R. Thom [20] and D. Sullivan [18], and

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[1]) Added on proof: *Topology 16* (1977), pp. 285-298.