ON THE GELFAND-FUKS COHOMOLOGY

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In this talk, we would like to report on the work of Gelfand and Fuks on the cohomology of the Lie algebra L_M of smooth vector fields on a manifold M, as well as on more recent developments, some of them obtained in collaboration with Raoul Bott.

1. DEFINITIONS

Gelfand-Fuks cohomology.

L_M will denote the Lie algebra of smooth vector fields on M, with the topology of uniform convergence of all derivatives on compact sets. For M compact, L_M can be thought as the Lie algebra of the group Diff_M of diffeomorphisms of M.

Gelfand and Fuks [7], have considered the differential graded algebra $C^*(L_M)$ of *continuous* multilinear alternate forms on L_M with values in R, the differential of a k-form f being the $(k+1)$-form df defined by

$$df(v_0, \ldots, v_k) = \sum_{o \leq r < s \leq k} (-1)^{r+s} f([v_r, v_s], v_0, \ldots, \hat{v}_r, \ldots, \hat{v}_s, \ldots, v_k)$$

where the v_i's are vector fields on M. So those cochains are like distributions.

Suppose that G is a Lie group acting smoothly and effectively on M. Then the Lie algebra $\mathfrak{g}$ of G is identified with a subalgebra of L_M. We shall denote by $C^*(L_M; G)$ the subalgebra of $C^*(L_M)$ of G-basic cochains, namely cochains invariant by G and which vanish if one of the argument v_i belongs to $\mathfrak{g}$.

The cohomology of $C^*(L_M)$ (resp. $C^*(L_M; G)$) will be denoted by $H^*(L_M)$ (resp. $H^*(L_M; G)$), and will be called the Gelfand-Fuks cohomology of M (resp. of M rel. to G).

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Models (Sullivan Theory) (cf. [18]).

$C^*(L_M)$ and $C^*(L_M; G)$ are examples of differential graded commutative (in the graded sense) algebras over **R**, abbreviated *DG*-algebras.

Among *DG*-algebras, we consider the equivalence relation generated by "$A \sim B$" if there is a morphism $\varphi: A \to B$ of *DG*-algebras inducing an isomorphism on cohomology. We say that M is a *model* for A if M is equivalent to A under this equivalence relation. Following the terminology of Sullivan, we say that M is a *minimal model* for A $\big($assuming $H^0(A) = R$ and $H^1(A) = 0\big)$ if M is a *free* algebra (namely the tensor product of a polynomial algebra on even dimensional generators by an exterior algebra on odd dimensional generators), the differential of each generator being decomposable (we also assume that generators are of degree bigger than one). The free algebra on a set of generators x_α will be denoted by $\Lambda(x_\alpha)$.

There is a contravariant functor from the category of topological spaces to the category of *DG*-algebras associating to the space X the *DG*-algebra $A^*(X)$ of real polynomial forms on its singular complex. If one takes instead rational polynomial forms, this functor induces an equivalence between rational homotopy types of 1-connected spaces with finite dimensional cohomology and equivalence classes of 1-connected *DG* algebras over Q with finite dimensional cohomology. A minimal model corresponds to a Postnikov decomposition. In particular the vector space of generators in the minimal model is the dual of the graded vector space $\pi_*(X) \otimes R$, where $\pi_i(X)$ is the i-th homotopy group of X.

We shall say that a *DG*-algebra A is a model for the space X if it is a model for the *DG*-algebra $A^*(X)$.

The main problem is to find good models for the *DG*-algebras $C^*(L_M)$ or $C^*(L_M; G)$, if possible finite dimensional in each degree.

As an example computed by Gelfand and Fuks [6], consider the case of the circle S^1. Then $H^*(L_{S^1})$ is the free algebra on generators u and v of degree 2 and 3 represented by the cocycles

$$u(f,g) = \int_0^1 \begin{vmatrix} f' & f'' \\ g' & g'' \end{vmatrix} dx \quad \text{and} \quad v(f,g,h) = \int \begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix} dx$$

where the vector fields on S^1 are identified with functions of period 1 on R. This is also a model for $C^*(L_{S^1})$.

If G is the group SO_2 of rotations of S^1, then $H(L_{S^1}; SO_2)$ is a model for $C^*(L_{S^1}; SO_2)$. It is generated by u and by an element e of degree 2 represented by

$$e(f,g) = \int_0^1 \begin{vmatrix} f, & g \\ f', & g' \end{vmatrix} dx$$

The only relation is $e\,u = 0$.

2. Connection with foliations

Let me indicate very briefly the relation with characteristic classes of flat bundles (cf. [12]).

$H^*(L_M, G)$ could also be interpreted as the differentiable cohomology of a suitable differentiable category (for more informations see [4] and [15]).

We consider on the product $X \times M$ of a smooth manifold X with M a smooth foliation F whose leaves have the same dimension as X and cut each fibers $\{x\} \times M$ transversally.

To such a foliation is naturally associated a continuous DG-algebra map

$$\chi_F : C^*(L_M) \to \Omega_X$$

where Ω_X is the DG-algebra of differential forms on X. In fact there is a bijection between such morphisms and foliations F as above.

Passing to cohomology, we get the characteristic map

$$H^*(L_M) \to H^*(X; R)$$

If we replace the trivial bundle by a bundle E with fiber M, base space X and structural group G, then for a foliation F on E complementary to the fibers, we still get a morphism

$$\chi_F : C^*(L_M; G) \to \Omega_X$$

hence a characteristic homomorphism

$$H^*(L_M, G) \to H^*(X; R)$$

Denoting by BG the classifying space for G-bundles, we also have the usual characteristic map $H^*(BG; R) \to H^*(X; R)$. This map factorizes through a map $H^*(BG; R) \to H^*(L_M; G)$ so that we get a commutative diagram

$$\begin{tikzcd} & H^*(L_M; G) \arrow[d] \\ H^*(BG; R) \arrow[ur] \arrow[dr] & \\ & H^*(X; R) \end{tikzcd}$$

So it is important to compute the map $H^*(BG; R) \to H^*(L_M; G)$. When G is a compact connected Lie group, then $H^*(BG; R)$ is the algebra $I(G)$ of invariant polynomials on the Lie algebra of G, and the map from $I(G)$ to $C^*(L_M; G)$ is given by a G-connexion in $C^*(L_M)$ (cf. [5]).

In the example above, namely $M = S^1$ and $G = SO_2$, then $H^*(BSO_2)$ is a polynomial algebra in a generator of degree 2, the Euler class, which is mapped on a non zero multiple of e.

3. The formal vector fields and the diagonal complex

Given a point x on M, we can consider the Lie algebra L_M^x of infinite jets at x of vector fields on M with the quotient topology. It is isomorphic to the Lie algebra $\mathfrak{a}_n$ of formal vector fields $\sum v_i(x)\, \partial/\partial x^i$ in R^n, where the $v_i(x)$ are formal power series in the coordinates $x^1, \ldots, x^n$.

The natural map $L_M \to L_M^x$ associating to a vector field its jet at x gives a DG-algebra morphism

$$C^*(L_M^x) \to C^*(L_M)$$

where $C^*(L_M^x)$ is the algebra of multilinear alternate forms on L_M^x depending only on finite order jets.

The first and most important step in the work of Gelfand-Fuks was the complete determination of the cohomology $H^*(\mathfrak{a}_n)$ of the topological Lie algebra of formal vector fields on R^n.

Theorem 1. (Gelfand-Fuks [8], [9]). *Let* $E(h_1, \ldots, h_n)$ *be the exterior algebra on generators* h_i *of degree* $2i-1$ *and let* $R[c_1, \ldots, c_n]^{\wedge}_{2n}$ *be the quotient of the polynomial algebra in generators* c_i *of degree* $2i$ *by the ideal of elements of degree* $> 2n$.

Then a model for $C^*(\mathfrak{a}_n)$ *is the DG-algebra*

$$WU_n = E(h_1, \ldots, h_n) \otimes R[c_1, \ldots, c_n]\,\hat{}_{2n}$$

with $dh_i = c_i$ *and* $dc_i = 0$.

It follows that $H^i(\mathfrak{a}_n) = 0$ for $1 \leqslant i \leqslant 2n$ and $i > n^2 + 2n$. Also the multiplicative structure is trivial; more precisely, WU_n is a model for a wedge of spheres (for instance S^3 for $n = 1$, $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ for $n = 2$) (cf. Vey [9]).

WU_n is also a model for the space F_n obtained by taking the restriction of the U_n-universal bundle over the $2n$-skeleton of its base space BU_n (cf. Gelfand-Fuks [8]). Note that this representation is compatible with the natural actions of $O_n \subset U_n$.

One can also consider the relative complex $C^*(\mathfrak{a}_n, O_n)$ or $C^*(\mathfrak{a}_n, SO_n)$ of O_n or SO_n-basic elements in $C^*(\mathfrak{a}_n)$, where O_n is the orthogonal group acting in the usual way on R^n, hence on $\mathfrak{a}_n$.

Define WO_n as the subalgebra of WU_n generated by the h_i with i odd and all the c_i. From theorem 1, it is easy to deduce the

THEOREM 1′ [12]. *WO_n is a model for $C^*(\mathfrak{a}_n, O_n)$.*
A model for $C^*(\mathfrak{a}_n, SO_n)$ is WO_n for n odd and

$$WSO_n = WO_n \otimes R[e] / (e^2 - c_n)$$

for n even, where $\deg e = n$ *and* $de = 0$.

From the finite dimensionality of $H^*(\mathfrak{a}_n)$, using a suitable spectral sequence, Gelfand and Fuks prove in particular [7].

THEOREM 2. *If $H^*(M)$ is finite dimensional, then $H^*(L_M)$ is finite dimensional in each degree.*

The Guillemin-Losik double complex.

First define $C^*(L_M, \Omega_M)$ as the algebra of continuous alternate multilinear forms on L_M with values in the algebra Ω_M of differential forms on M. We have two differentials, the first one defined as in 1 and the second one by the exterior differential in Ω_M. So this is a double complex and we can consider the associated total differential.

$C^*_\Delta(L_M, \Omega_M)$ is the subcomplex of $C^*(L_M, \Omega_M)$ of those forms associating to a sequence $v_1, \ldots, v_k$ of vector fields on M a differential form $f(v_1, \ldots, v_k)$ whose value at $x \in M$ depends only on finite order jets of the v_i's at x.

THEOREM 3. (Guillemin [10], Losik [17]). *$C^*_{\triangle}(L_M, \Omega_M)$ is a model for a bundle E with fiber F_n, base space M, associated to the tangent bundle of M.*

More precisely, a model for $C^*_{\triangle}(L_M, \Omega_M)$ is the DG-algebra $\Omega_M \otimes WU_n$ over Ω_M, where

$$d(1 \otimes c_i) = 0 \quad d(1 \otimes h_i) = 1 \otimes c_i - p_{i/2} \otimes 1$$

where $p_{i/2}$ is zero if i is odd and is a form representing the Pontrjagin class of M of degree $2i$ if i is even.

Note that if a foliation F on $X \times M$ transverse to the fibers $\{x\} \times M$ is given, one has a characteristic homomorphism

$$C^*(L_M, \Omega_M) \to \Omega_{X \times M}$$

One has also a morphism

$$WO_n \to C^*_{\triangle}(L_M, \Omega_M)$$

(or $WU_n \to C^*(L_M, \Omega_M)$ in case M has trivial Pontrjagin classes) whose composition with the previous one is the usual characteristic homomorphism for the foliation F (cf. [3], [12]).

4. MAIN THEOREM

THEOREM 1. *$C^*(L_M)$ is a model for the space Γ of continuous sections of the bundle E described in the theorem above.*

This result, first conjectured by Bott (and also Fuks), has been proved by several people (Bott-Segal[1]), Fuks-Segal, Haefliger [13], Ph. Trauber, and others).

Suppose that G is a compact connected Lie group acting on M. Then it also acts on the bundle E and on its space of sections. Let us denote by Γ_G the total space of the bundle with fiber Γ associated to the universal G-bundle with base space BG.

THEOREM 1′. *$C^*(L_M; G)$ is a model for the space Γ_G.*

The way I proved theorem 1 was to construct first a tentative algebraic model A for Γ following ideas of R. Thom [20] and D. Sullivan [18], and

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[1]) Added on proof: *Topology 16* (1977), pp. 285-298.

a morphism of A in $C^*(L_M)$. Then one proves directly that it induces an isomorphism in cohomology. The fact that A is also a model for Γ was proved in a similar way (cf. [14]).

When M has a finite dimensional model, one can construct a model for Γ which is finite dimensional in each degree, and with it one can make explicit calculations.

Note that the inclusion $C^*_\triangle(L_M, \Omega_M) \to C^*(L_M, \Omega_M)$ is a model for the evaluation map $\Gamma \times M \to E$ associating to a section s and a point x of M the element $s(x)$ of E.

For computations along the lines of the spectral sequence of Gelfand-Fuks, see Cohen and Taylor [22].

The proof of theorem 1′ is very similar to the proof of theorem 1. In the next paragraph, we explain the construction of an algebraic model for Γ_G suitable for computations. In § 6, we indicate briefly why this is a model for Γ_G.

5. Construction of an algebraic model for the space of sections of a fiber bundle ([20], [18], [13]).

As a guide, consider first the geometric situation. Let $p: E \to M$ be a fiber bundle with base space M, fiber F and let Γ be the space of continuous sections of E.

We have the commutative diagramm

1)
$$\begin{tikzcd} M \times \Gamma \arrow[rr, "e"] \arrow[d] \arrow[dr] & & E \arrow[dl, "p"] \\ \Gamma \arrow[dr] & M \arrow[d] & \\ & * & \end{tikzcd}$$

where e is the evaluation map associating to the point x of M and the section s the point $s(x)$ of E. The other maps are natural projections ($*$ is a point).

Suppose that a topological group G acts on M and also on E in a way compatible with p. Then G acts also on Γ, and all the maps in the diagramm are equivariant.

For a space X on which G acts, let us denote by X_G the bundle with fiber X associated to the principal universal G-bundle P with base space BG $(= *_G)$.

From 1) we get the corresponding commutative diagramm

2)
$$\begin{tikzcd}
(M \times \Gamma)_G \arrow[rr] \arrow[d] \arrow[dr] & & E_G \arrow[dl] \\
\Gamma_G \arrow[dr] & M_G \arrow[d] & \\
& BG &
\end{tikzcd}$$

We try now to construct an algebraic analogue of this diagramm. *We assume that the connectivity of the fiber F of E is bigger than the dimension n of M.*

Choose a DG-algebra B which is a model of BG and assume that *we can represent the bundle M_G by a DG-algebra A, the projection being represented by a morphism $B \to A$, and such that A, as a module over B, is free and finite dimensional with a basis $s_1, \ldots, s_k$, where the degree of s_i is not bigger than n* (see examples below).

Then we construct the Postnikov decomposition (or minimal model) of the bundle $E_G \to M_G$. Algebraically, this means that we take a model for E_G which is a tensor product $A \otimes \Lambda(x_\alpha)$, where $\Lambda(x_\alpha)$ is a free graded algebra on an ordered set of generators x_α, the differential of each x_α, being in the subalgebra generated by A and the preceding x_β. Of course the natural inclusion of A in $A \otimes \Lambda(x_\alpha)$ has to be a model for the projection $E_G \to M_G$. Such a model, with a finite number of generators x_α in each degree, always exists if F is 1-connected and with finite dimensional cohomology, and if G is a connected Lie group (cf. [13], [18]).

A model for Γ_G will be the algebra $B \otimes \Lambda(x^i_\alpha)$, where $\Lambda(x^i_\alpha)$ is the free algebra on generators x^i_α, $i = 1, \ldots, k$, and $\deg x^i_\alpha = \deg x_\alpha - \deg s^i$. By our assumptions, $\deg x^i_\alpha > 0$.

A model for the map e will be the morphism

$$\varepsilon : A \otimes \Lambda(x_\alpha) \to A \otimes \Lambda(x^i_\alpha)$$

of A-algebras defined by

$$\varepsilon(1 \otimes x_\alpha) = \sum_i s^i \otimes x'^i_\alpha .$$

The differential on $B \otimes \Lambda(x^i)$ is then uniquely defined by the conditions that $B \otimes \Lambda(x^i)$ should be a DG-algebra over B and that ε should commute with the differential given by the isomorphism with $A \otimes_B (B \otimes \Lambda(x'^i_\alpha))$.

The algebraic analogue of diagramm 2) is the commutative diagramm of DG-algebras

$$\begin{tikzcd}
A \otimes_B (B \otimes \Lambda(x'^i_\alpha)) & & A \otimes \Lambda(x_\alpha) \arrow[ll] \\
B \otimes \Lambda(x'^i_\alpha) \arrow[u] & A \arrow[ul] \arrow[ur] & \\
& B \arrow[u] \arrow[ul] &
\end{tikzcd} \tag{2)}$$

Examples.

1. For M, take the 2-sphere S^2 and for E the trivial bundle $S^2 \times S^4$, so that Γ is the space of continuous maps of S^2 in S^4. The group G will be the rotation group SO_3 acting on S^2 as usual and trivialy on S^4.

As model B for BG we take the polynomial algebra $R[p_1]$ in a generator p_1 of degree 4. A model for M_G is the algebra A quotient of the polynomial algebra $\Lambda(s, p_1)$, where $\deg s = 2$, by the ideal generated by $s^2 - p_1$. The differential is zero. The elements 1 and s form a basis for the B-module A.

A minimal model for the bundle E_G is $A \otimes \Lambda(x, y)$, where $\Lambda(x, y)$ is the free algebra with generators x of degree 4, and y of degree 7, and $dy = x^2$.

According to the preceding recipe, a model for Γ_G is the algebra $R[p_1] \otimes \Lambda(x, y, \bar{x}, \bar{y})$ with $\deg\bar{x} = 2$, $\deg\bar{y} = 5$, the image of x by ε being $1 \otimes x + s \otimes \bar{x}$, similarly for y. The differential is given by $dx = d\bar{x} = 0$, $dy = x^2 + p_1\bar{x}^2$, $d\bar{y} = 2x\bar{x}$.

2. Take M as the circle, E as the product $S^1 \times F$, where F is a simply connected space, so that Γ is just the space of continuous maps of S^1 in F (case studied by Sullivan [19]). For G we take the group of rotations of the circle, acting trivially on F.

Represent F by its minimal model $\Lambda(x_\alpha)$. A model B for BG is the polynomial algebra $R[e]$ in a generator e of degree 2 and a model A for M_G

is the free commutative algebra $\Lambda(s, e)$, where $\deg s = 1$ and $ds = e$. As a B-module, it is free with basis 1 and s. A model for E_G is just $A \otimes \Lambda(x_\alpha)$.

As model for Γ_G, we take $R[e] \otimes \Lambda(x_\alpha, \bar{x}_\alpha)$, where $\deg \bar{x}_\alpha = \deg x_\alpha - 1$, the image of x_α by ε being $1 \otimes x_\alpha + s \otimes \bar{x}_\alpha$. The differential d is described as follows (compare with Sullivan [18] or [19]). Let h be the derivation of degree -1 of $\Lambda(x_\alpha, \bar{x}_\alpha)$ given by $hx_\alpha = \bar{x}_\alpha$ and $h\bar{x}_\alpha = 0$. Then if d_0 denotes the differential in $\Lambda(x_\alpha)$ identified to a subalgebra of $\Lambda(x_\alpha, \bar{x}_\alpha)$, we have

$$de = 0, dx_\alpha = d_0 x_\alpha - e\,\bar{x}_\alpha, d\bar{x}_\alpha = -hd_0 x_\alpha$$

Remark. In the case where E is the bundle described in § 4, its minimal model $A \otimes \Lambda(x_\alpha)$ over M_G is complicated, because there is an infinite number of generators x_α (except for $n = 1$) labelled by a basis of the rational homotopy of a wedge of spheres, so by a basis of the free graded Lie algebra $L(n)$ generated by the spheres of this wedge (cf. [13]).

6. Sketch of the proof of the main theorem and applications

We represent the universal principal G-bundle as a limit of finite dimensional bundles P_k and we denote by Ω_P the inverse limit of algebras of forms Ω_{P_k}.

First note that we can replace $C^*(L_M; G)$ by the DG-algebra $C^*(L_M, \Omega_P)_G$ of G-basic elements in $C^*(L_M, \Omega_P)$ (compare with Cartan [5], exposé 20).

A model for E_G will be the algebra $C^*_\Delta(L_M, \Omega_{M \times P})_G = [C^*_\Delta(L_M, \Omega_M \hat{\otimes} \Omega_P]_G$ and a model for the evaluation map will be the inclusion of this DG-algebra in $C^*(L_M, \Omega_{M \times P})_G$.

In the construction of § 5, we choose $B = \Omega_{BG}$ as model for BG and, instead of taking for A a finite dimensional module over B, we take the DG-algebra $\Omega_{M_G} \approx [\Omega_{M \times P}]_G$ as model for M_G. We have to build the model for Γ_G along the same lines as in § 5, but in more intrinsic terms like in [13]. The minimal model (or Postnikov decomposition of E_G) will be of the form $A \otimes S^*(V)$, where $S^*(V)$ denotes the algebra of symmetric multilinear forms on a graded vector space V (cf. [13]).

As an algebra, the model for Γ_G will be the algebra $S^*_B(A \otimes V, B)$ of continuous symmetric B-multilinear forms on the graded B-module $A \otimes V$. One can construct a map of this model in $C^*(L_M, \Omega_{M \times P})_G$ and prove that it induces an isomorphism in cohomology.

Similarly, one can prove that $S_B^*(A \otimes V, B)$ is effectively a model for the space of sections Γ_G (cf. [14]).

Eventually for computations, one proves that one gets also a model for Γ_G by using instead of Ω_{M_G} a DG-algebra A as in § 5 which is a finite dimensional free B-module.

7. Example of a computation

Let us consider the case where M is the n-sphere S^n, G the rotation group SO_{n+1} and E the bundle described in § 3. A model for M_G is the DG-algebra A defined by

$$A = R[p_1, \dots, p_k, s] / (s^2 - p_k) \qquad d \equiv 0 \qquad \text{for} \quad n = 2k$$

$$\text{or} \qquad A = R[p_1, \dots, p_{k-1}, \chi] \otimes E(s) \quad ds = \chi \quad \text{for} \quad n = 2k-1$$

where $\deg p_i = 4i$ and $\deg s = n$.

A model for E_G is obtained by taking the tensor product of A with WU_n, the differential being defined by

$$dh_i = c_i - p_{i/2} \quad \text{and} \quad dc_i = 0 .$$

By the way, WSO_n is also a model for E_G.

We now consider the case $n = 2$. The minimal model of E_G is the DG-algebra which begins as

$$A \otimes \Lambda(x_1, x_2, x_3, x_4, x_5, x_{12}, x_{13}, x_{23}, \dots)$$

where

$$\deg x_1 = \deg x_2 = 5, \deg x_3 = 7, \deg x_4 = \deg x_5 = 8 ,$$
$$\deg x_{12} = 9, \deg x_{13} = \deg x_{23} = 11 ,$$

etc.

(there is an infinite number of generators).

The differential is defined by

$$dx_1 = dx_2 = 0, dx_3 = -p_l^2, dx_4 = p_1 x_1, dx_5 = p_1 x_2 ,$$
$$dx_{12} = x_1 x_2, dx_{13} = x_1 x_3 - p_1 x_4, dx_{23} = x_2 x_3 - p_1 x_5 ,$$

etc.

According to the construction of § 5, a minimal model for the bundle $\Gamma_G \to B_G$ begins as

$$R[p_1] \otimes \Lambda(\bar{x}_1, \bar{x}_2, x_1, x_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, x_3, \bar{x}_{12}, x_4, x_5, \dots)$$

where

$$\deg \bar{x}_1 = \deg x_i - 2, \varepsilon(x_i) = 1 \otimes x_i + s \otimes \bar{x}_i,$$

dx_i is as above and $dx_{12} = x_1 x_2 + p_1 \bar{x}_1 \bar{x}_2$

$$d\bar{x}_1 = d\bar{x}_2 = d\bar{x}_3 = 0, \; d\bar{x}_4 = p_1 \bar{x}_1, d\bar{x}_5 = p_1 \bar{x}_2,$$
$$d\bar{x}_{12} = \bar{x}_1 x_2 + x_1 \bar{x}_2,$$

etc.

A basis for $H^*(\Gamma_G) = H^*(L_{S^2}, SO_3)$ is given by the classes of the cocycles

$$\bar{x}_1, \bar{x}_2, p_1, x_1, x_2, \bar{x}_3, \bar{x}_1\bar{x}_2, x_1\bar{x}_1, x_1\bar{x}_2, x_2\bar{x}_2, \bar{x}_1\bar{x}_3,$$
$$\bar{x}_2\bar{x}_3, \bar{x}_1\bar{x}_4, \bar{x}_2\bar{x}_5, \bar{x}_1\bar{x}_5 + \bar{x}_2\bar{x}_4, p_1\bar{x}_3 ,$$

etc.

The first multiplicative relations are

$$p_1\bar{x}_1 \sim 0, p_1\bar{x}_2 \sim 0, \bar{x}_1 x_2 \sim \bar{x}_2 x_1, p_1^2 \sim 0, \text{ etc.}$$

The first "exotic" class is given by the cocycle $\bar{x}_1\bar{x}_2\bar{x}_{12}$ of degree 13.

The classes $\bar{x}_1$ and $\bar{x}_2$ correspond to the classes described by Raoul in his lecture [4], for $n = 2$.

We now give an example of a general statement

THEOREM. *The kernel of the map*

$$H^*(BSO_{n+1}) \to H^*(L_{S^n}, SO_{n+1})$$

is the ideal generated by the elements which are polynomials of *degree* $> 2n$ *in the Pontrjagin classes* $p_1, \ldots, p_{[n/2]}$.

As a consequence, we get exactly what is implied by the vanishing theorem of Bott [1]. For instance, for n odd, the image of the powers of the Euler class is non zero. So one can ask for examples of flat $(2k+1)$-sphere bundles with a non zero power of the Euler class.

One can also check that the homomorphism (see end of § 3)

$$WSO_n \to C^*(L_{S^n}, SO_{n+1}, \Omega_{S^n})$$

induces an injection in cohomology.

8. Case of a manifold with boundary

More generally we consider a closed manifold N of dimension p in a manifold M of dimension n. $L_{M,N}$ will denote the subalgebra of L_M of those vector fields on M which are tangent to N. An interesting particular case is when N is the boundary ∂M of M. For M compact, $L_{M,\partial M}$ can be considered as the Lie algebra of the group of diffeomorphisms of M.

First we consider the formal vector fields. *Let* $\mathfrak{a}_{n,p}$ *be the Lie subalgebra of formal vector fields on* R^n *which are tangent to* R^p identified to a linear subspace of R^n. Again $C^*(\mathfrak{a}_{n,p})$ denotes the DG-algebra of those multilinear alternate forms on $\mathfrak{a}_{n,p}$ depending only on finite order jets.

We describe a finite dimensional model for $C^*(\mathfrak{a}_{n,p})$. Let $E(h'_1, \ldots, h'_p, h''_1, \ldots, h''_{n-p})$ be the exterior algebra in generators h'_i and h''_j of degree $2i-1$. Let $R[c'_1, \ldots, c'_p, c_1'', \ldots, c''_{n-p}]^{\wedge}_{2p}$ be the quotient of the polynomial algebra in generators c'_i and c_i'' of degree $2i$ by the ideal of elements of degree $> 2p$.

Define

$$\begin{aligned} W U_{n,p} &= E(h'_1, \ldots, h'_p, h''_1, \ldots, h''_{n-p}) \\ &\otimes R[c'_1, \ldots, c'_p, c''_1, \ldots, c''_{n-p}]^{\wedge}_{2p} \end{aligned}$$

as the DG-algebra with differential defined by

$$dh_i' = c_i', \quad dh_i'' = c_i'', \quad dc_i' = 0, \quad dc_i'' = 0 .$$

This is a model for the space $F_{n,p}$ obtained by restricting the universal principal $(U_p \times U_{n-p})$-bundle over the $2p$-skeleton of its basis represented by a product of Grassmanians with the usual even dimensional cell decomposition.

If $n \leqslant 2p$, $WU_{n,p}$ is also a model for a wedge of spheres. When $n > 2p$, it is a model for the product of the wedge of spheres corresponding to $WU_{2p,p}$ by $S^{2p+1} \times S^{2p+3} \ldots \times S^{2n-2p-1}$.

Theorem 1 (Koszul [11]). *There is a natural morphism*

$$WU_{n,p} \to C^*(\mathfrak{a}_{n,p})$$

inducing an isomorphism in cohomology.

As a consequence, $H^i(\mathfrak{a}_{n,p}) = 0$ for $0 < i \leqslant 2p$ and $i > p^2 + (n-p)^2 + 2p$. When $n \leqslant 2p$, the multiplication is trivial.

To have a model for the homomorphism induced by the inclusion of $\mathfrak{a}_{n,p}$ in $\mathfrak{a}_n$, we have the commutative diagramm

$$\begin{array}{ccc} C^*(\mathfrak{a}_n) & \longrightarrow & C^*(\mathfrak{a}_{n,p}) \\ \uparrow & & \uparrow \\ WU_n & \longleftarrow & WU_{n,p} \end{array}$$

where the second horizontal map sends h_i on $h_i' + h_i''$ and c_i on $c_i' + c_i''$ (by convention, h_i' or h_i'' is zero for $i > p$ or $i > n\text{-}p$, idem for c_i' and c_i''). Note that the natural map of theorem 1 should map the c_i' *s* and c_i'' not on the usual Chern classes defined by the connection but on the polynomials in Chern classes corresponding to $\sum x_k^i$, the Chern classes being the elementary symmetric functions in the formal variables x_k. These horizontal maps are also models for an inclusion of $F_{n,p}$ in F_n.

We consider again the bundle E over M associated to the tangent bundle of M and with fiber F_n. Its restriction above N contains a subbundle E' with fiber $F_{n,p}$.

THEOREM. *$C^*(L_{M,N})$ is a model for the space $\Gamma_{M,N}$ of continuous sections of the bundle E whose restriction to N have values in the subbundle E'.*

To make explicit computations, we construct a model for $\Gamma_{M,N}$, which will be finite dimensional in each degree when M and N have finite dimensional models. This is the purpose of the next paragraph.

9. Construction of a model for $C^*(L_{M,N})$

Consider the commutative diagramm of Lie algebras

$$\begin{array}{ccc} L_{M,N} & \longrightarrow & L_M \\ \downarrow & & \downarrow \\ L'_{M,N} & \longrightarrow & L'_M \end{array}$$

where L'_M and $L'_{M,N}$ are the quotients of L_M and $L_{M,N}$ by the subalgebra L_M^0 of vector fields on M whose infinite jet vanish at points of N.

The corresponding geometric diagramm is

$$\begin{array}{ccc} \Gamma_{M,N} & \longrightarrow & \Gamma_M \\ \Big\downarrow & & \Big\downarrow \pi \\ \Gamma'_{M,N} & \longrightarrow & \Gamma'_M \end{array}$$

where Γ'_M denotes the space of sections of E restricted to N and $\Gamma'_{M,N}$ the space of sections of E'. The vertical maps associate to a section its restriction above N.

π is a fibration and $\Gamma_{M,N}$ is the fiber product of Γ_M and $\Gamma'_{M,N}$ over Γ'_M.

The spectral sequence of the fibration $\Gamma_{M,N} \to \Gamma'_{M,N}$ will correspond to the Hochschild-Serre spectral sequence [16] associated to the ideal L^0_M in $L_{M,N}$ (using continuous cochains). The DG-algebra $C^* (L^0_M)$ will be a model for the fiber.

We assume that we can represent the inclusion of N *in* M *by a surjection* $r\colon A \to B$ *of DG-algebras which are finite dimensional and such that* $A^i = 0$ *for* $i > n = \dim M$ *and* $B^i = 0$ *for* $i > p = \dim N$.

This is possible in particular if M and N are simply connected with finite dimensional real cohomology.

Let $a_1, \ldots, a_s, b_1, \ldots, b_t$ be a basis of A such that the $a'_i s$ form a basis of the kernel $\bar{A}$ of r. Hence the $r\,(b_j)$'s form a basis of B.

Let $\Lambda\,(x_\alpha)$ (resp. $\Lambda\,(y_\lambda)$) be a minimal model for F_n (resp. $F_{n,p}$), or equivalently of WU_n (resp. $WU_{n,p}$). Then the bundle E (resp. E') has a minimal model of the form $A \otimes \Lambda\,(x_\alpha)$ (resp. $B \otimes \Lambda\,(y_\lambda)$), where the differential is twisted by terms depending on the choice of representatives for the Pontrjagin classes of M (cf. [13]).

A model for $\Gamma_{M,N}$ will be the free algebra $\Lambda\,(x^i{}_\alpha, y^j{}_\lambda)$ on generators x^i_α, $i = 1, \ldots, s$, and $y^j{}_\lambda$, $j = 1, \ldots, t$, $\deg x^i_\alpha = \deg x - \deg a_i$, $\deg y^j{}_\lambda = \deg y_\lambda - \deg b_j$.

To get the differential, we proceed as follows. Recall that a model for Γ_M is the algebra $\Lambda\,(x^i{}_\alpha, z^j{}_\alpha)$, $\deg z^j_\alpha = \deg x_\alpha - \deg b_j$, with a suitable differential (cf. [18], [13] or § 5 with G the identity). Also models for $\Gamma'_{M,N}$ and Γ'_M are of the form $\Lambda\,(y^j{}_\lambda)$ and $\Lambda\,(z^i{}_\alpha)$, resp. with suitable differentials. One has DG-algebra maps

$$\Lambda\,(z^j{}_\alpha) \to \Lambda\,(x^j{}_\alpha, z^j{}_\alpha)$$
$$\Lambda\,(z^j{}_\alpha) \to \Lambda\,(y^j{}_\lambda)$$

which are models for the maps $\Gamma_M \to \Gamma'_M$ and $\Gamma'_{M,N} \to \Gamma'_M$. The first one is obvious and the second one is completely characterized by the map $WU_n \to WU_{n,p}$.

Now we get the differential on $\Lambda(x^j_\alpha, y^j_\lambda)$ by considering this algebra as the tensor product over $\Lambda(z^j_\alpha)$ of $\Lambda(x^j_\alpha, z^j_\lambda)$ with $\Lambda(y^j_\lambda)$.

One can make a similar construction using for A and B the DG-algebras Ω_M and Ω_N of differential forms on M and N. Of course one has to work again in more intrisic terms and use the C^∞-topology on Ω_M and Ω_N (compare with [13]). In this way one gets a DG-algebra which is also a model for $\Gamma_{M,N}$ (in fact one proves directly that it is a model for the DG-algebra constructed above), with a map in $C^*(L_{M,N})$ inducing an isomorphism in cohomology.

Summing up, we get the following result.

THEOREM. *Assume that the inclusion of N in M has a model which is a surjection of finite dimensional DG-algebras. One can construct explicitely a model for $C^*(L_{M,N})$ which is finite dimensional in each degree.*

Example. Suppose that M is the disk D^2 and N its boundary $\partial D^2 = S^1$. As the inclusion of $F_{2,1}$ in F_2 is homotopically trivial (equivalently the morphism $WU_2 \to WU_{2,1}$ is homotopic to zero), the bundle $\Gamma_{M,N} \to \Gamma'_{M,N}$ is trivial. WU_2 is a model for $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$ and $WU_{2,1}$ for $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$.

Hence $C^*(L_{D^2, \partial D^2})$ is a model for the space which is the product of the space of maps of S^1 in $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$ with the second loop space of $S^5 \vee S^5 \vee S^7 \vee S^8 \vee S^8$.

One can write down quite explicitely the minimal model for that space, but it is harder to compute the cohomology of the first factor. It has an infinite number of multiplicative generators.

10. SOME OTHER PROBLEMS

1. As coefficient for the Gelfand-Fuks cochains, one might consider, instead of the field R with the trivial action of L_M, a topological L_M-algebra A. The problem is to find a model for the DG-algebra $C^*(L_M, A)$ of continuous multilinear alternate forms on L_M with values in A. The differential is defined by the usual formula involving the action of L_M on A.

For that case, results similar to the one mentionned in this report have been obtained by Fuks-Segal (unpublished) and by T. Tsujishita [21].

For instance, when A is the algebra of smooth functions on M on which L_M acts by Lie derivative, their result is as follows. As it is described in § 3, the bundle E over M has a fiber F_n which is itself a principal U_n-bundle. Let us fix a fiber $F_n^\circ \approx U_n$ of this bundle; as it is invariant by the structural group $O_n \subset U_n$ of E, we get a subbundle E_o of E with typical fiber F_n^o. Then $C^* (L_M, A)$ will be a model for the inverse image of E_o by the evaluation map $M \times \Gamma \to E$.

2. One of the most interesting problems is to know when, for a given class α in $H^* (L_M)$, there is a space X and a foliation F on $X \times M$ transverse to the fibers such that the image of α in $H^* (X)$ by the characteristic homomorphism (cf. 2) is non zero.

Very recent and remarkable results of Fuchs [23] show that this is the case for all classes coming from WSO_n. (For earlier partial results, see [4].) One might expect that his method will apply in general and show that the answer is affirmative for all classes in $H^* (L_M)$ (and also for the similar problem with $H^* (L_M; G)$).

There is also the problem of the possible continuous variations of characteristic classes for flat bundles which would be interesting to study (cf. [23]).

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