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§ 1. THE COINCIDENCE-FIXED-POINT (C.F.P.) INDEX

(1.1) Let $p: E \rightarrow B$ denote a euclidean neighborhood retract over B (abbrev. ENR_B), where B , and hence E , is an ENR. Altogether this means that $p: E \rightarrow B$ embeds as a neighborhood retract into the projection $\mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m$, for some m, n . We refer the reader to [2], §1, for the precise definitions but remark that every smooth submersion and every fibration (with base and total space ENR) qualifies for $p: E \rightarrow B$.

We consider continuous maps $g: D_g \rightarrow E$, $\varphi: D_\varphi \rightarrow B$, where D_g, D_φ are open subsets of E , and $pg = p|_{D_g}$ (i.e., g is fibre-preserving). We let $\text{Fix}(g) = \{x \in D_g \mid gx = x\}$ and $\text{Coinc}(\varphi, p) = \{x \in D_\varphi \mid \varphi x = px\}$, and we assume that $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ is compact. Under these circumstances we shall define an integer $J(g, \varphi) \in \mathbf{Z}$ which is akin to the Hopf fixed-point index. It "counts" the points in $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ in a weighted and homotopy-invariant fashion. It is the Hopf index of g resp. φ if B is a single point resp. p is the identity map of B .

(1.2) By definition [2], 1.1 of an ENR_B , we have that E is a fibre-preserving neighborhood retract of some $\mathbf{R}^n \times B$. In fact, for the present purpose we can use any product $Y \times B$, i.e. we'll use mappings $E \xrightarrow{i} V \xrightarrow{r} E$ such that $V \subset Y \times B$ is open, $ri = id$, and i, r are maps over B . In formulas,

$$(1.3) \quad ix = (i'x, px), \quad \text{where } i': E \rightarrow Y,$$

$$(1.4) \quad pr(y, b) = b, \quad \text{for } (y, b) \in V,$$

$$(1.5) \quad r(i'x, px) = x, \quad \text{for } x \in E.$$

Consider the following sequence of maps

$$(1.6) \quad D_g \cap D_\varphi \xrightarrow{(g, \varphi)} E \times B \xrightarrow{i' \times id} Y \times B \supset V \xrightarrow{r} E.$$

Its composite $[g, \varphi]$ is defined in $D_V = (i'g, \varphi)^{-1} V$ which is an open subset of $(D_g \cap D_\varphi)$, and hence of E . Thus

$$(1.7) \quad [g, \varphi]: D_V \rightarrow E, \quad [g, \varphi](x) = r(i'gx, \varphi x).$$

If $x \in D_g \cap \text{Coinc}(\varphi, p)$ then

$$(i'g, \varphi)x = (i'gx, px) = (i'gx, pgx) = igx \in V,$$

hence $[g, \varphi]x$ is defined and equals $rigx = gx$. It follows that $\text{Fix}(g) \cap \text{Coinc}(\varphi, p) \subset \text{Fix}[g, \varphi] = \{x \in D_V \mid [g, \varphi]x = x\}$. Conversely, x

$= [g, \varphi] x$ implies $px = p[g, \varphi] x = pr(i'gx, \varphi x) = \varphi x$, hence $x \in \text{Coinc}(\varphi, p)$, and $gx = [g, \varphi] x = x$. Altogether

$$(1.8) \quad \text{Fix}(g) \cap \text{Coinc}(\varphi, p) = \text{Fix}[g, \varphi].$$

In particular, $[g, \varphi]: D_V \rightarrow E$ has a compact fixed-point set, and we can assign to it its Hopf-index $I[g, \varphi] \in \mathbb{Z}$ — for instance as in [1], VII,5.10. Furthermore,

(1.9) PROPOSITION AND DEFINITION. *The Hopf-index $I[g, \varphi] \in \mathbb{Z}$ depends only on (g, φ) , not on the choice of the neighborhood retraction i, r . We denote this integer by $J(g, \varphi)$, and call it the c.f.p.-index of (g, φ) ; thus $J(g, \varphi) = I[g, \varphi]$.*

Proof. Because the range B of the maps φ, p is ENR, these two maps are homotopic in a neighborhood of $\text{Coinc}(\varphi, p)$. In fact (cf. [1], IV,8.6), there is an open neighborhood U of $\text{Coinc}(\varphi, p)$ in D_φ , and a deformation $\vartheta_t: U \rightarrow B$, $0 \leq t \leq 1$, such that

$$(1.10) \quad \vartheta_0 = p|U, \vartheta_1 = \varphi|U, \vartheta_t x = px \text{ for } x \in \text{Coinc}(\varphi, p) \text{ and all } t.$$

Consider then two neighborhood retractions

$$\begin{aligned} E &\xrightarrow{i} V \xrightarrow{r} E, \quad V \subset Y \times B; \quad ix = (i'x, px), \\ E &\xrightarrow{j} W \xrightarrow{s} E, \quad W \subset Z \times B; \quad jx = (j'x, px), \end{aligned}$$

as above, and the corresponding maps $[g, \varphi]_1, [g, \varphi]_2$ as defined by 1.6. We have to show $I([g, \varphi]_1) = I([g, \varphi]_2)$. In order to do so we can (cf. [1], VII,5.11) restrict attention to an arbitrary open neighborhood N of $\text{Fix}([g, \varphi]_i) = \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$. And we shall show that $[g, \varphi]_i|N$ are homotopic ($i=1, 2$) without moving the fixed point set, provided N is sufficiently small. The homotopy is given by the formula

$$(1.11) \quad \theta_t x = s(j'r(i'gx, \vartheta_t x), \varphi x).$$

This is defined for (x, t) such that $x \in D_g \cap U$, $v = (i'gx, \vartheta_t x) \in V$, and $w = (j'rv, \varphi x) \in W$; the set of all such (x, t) is an open subset D_θ of $E \times [0, 1]$. If $x \in \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ then

$$v = (i'gx, \vartheta_t x) = (i'x, px) = ix \in V, \text{ and } rv = x,$$

hence

$$w = (j'rv, \varphi x) = (j'x, px) = jx \in W, \text{ and } \theta_t x = sw = x.$$

Therefore, $(\text{Fix}(g) \cap \text{Coinc}(\varphi, p)) \times [0, 1] \subset D_\theta$, and $(\text{Fix}(g) \cap \text{Coinc}(\varphi, p)) \subset \text{Fix}(\theta_t)$ for all t . It follows that

$$N = \{x \in E \mid (x, t) \in D_\theta \text{ for all } t\}$$

is an open neighborhood of $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ in which the deformation θ is defined (by 1.11).

Suppose now $x \in N$ is a fixed point of θ_t , thus $x = s(j'r(i'gx, \vartheta_t x), \varphi x)$. Apply p , using 1.4 for s , and get $px = \varphi x =$, hence $\vartheta_t x = px$ by 1.10, hence $r(i'gx, \vartheta_t x) = r(i'gx, px) = r(i'gx, pgx) = rigx = gx$, hence $x = \theta_t x = s(j'gx, px) = s(j'gx, pgx) = sjgx = gx$; altogether, $x \in \text{Coinc}(\varphi, p) \cap \text{Fix}(g)$. It follows that the fixed point set $\text{Fix}(\theta_t) = \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ for all t . In particular, $\cup_{t \in [0, 1]} \text{Fix}(\theta_t)$ is compact, hence (cf. [1].VII, 5.15) all θ_t have the same Hopf-index $I(\theta_t)$. But $r(i'gx, \vartheta_0 x) = r(i'gx, px) = r(i'gx, pgx) = gx$, hence $\theta_0 x = s(j'gx, \varphi x) = [g, \varphi]_2 x$. To calculate θ_1 we first remark that $p[g, \varphi]_1 x = \varphi x$, by 1.7 and 1.4; also $r(i'gx, \vartheta_1 x) = r(i'gx, \varphi x) = [g, \varphi]_1 x$, hence $\theta_1 x = s(j'[g, \varphi]_1 x, p[g, \varphi]_1 x) = sj[g, \varphi]_1 x = [g, \varphi]_1 x$. $\square$

(1.12) *The product case* $E = F \times B$, $p = \text{projection}$. In this case $g: D_g \rightarrow F \times B$ has the form $g(y, b) = (\gamma(y, b), b)$ with $\gamma: D_g \rightarrow F$. The two maps (γ, φ) combine to a map $(\gamma, \varphi): D \rightarrow F \times B$, where $D (= D_g \cap D_\varphi)$ is an open subset of $F \times B$, and $\text{Fix}(\gamma, \varphi) = \text{Fix}(g) \cap \text{Coinc}(\varphi, p)$. In order to obtain the c.f.p.-index $J(g, \varphi)$ one can use $Y = F$ and the neighborhood retraction $i = r = \text{identity-map}$ of $Y \times B$. The definition 1.9 then shows that

$$J(g, \varphi) = I(\gamma, \varphi);$$

i.e. in the product case the c.f.p.-index of (g, φ) is simply the Hopf-index of $(y, b) \mapsto (\gamma(y, b), \varphi(y, b))$.

The procedure 1.6-1.9 in the general case, on the other hand, can be considered as a reduction to the product case.

(1.13) *General properties of $J(g, \varphi)$* follow from corresponding properties of the Hopf-index. For instance, $J(g, \varphi)$ is additive with respect to topological-sum decompositions of $\text{Fix}(g) \cap \text{Coinc}(g, \varphi)$, it is invariant under deformations such that $\cup_{0 \leq t \leq 1} \text{Fix}(g_t) \cap \text{Coinc}(\varphi_t, p)$ is compact, it

depends only on the germ of (g, φ) around $\text{Fix}(g) \cap \text{Coinc}(\varphi, p)$ — in particular, $J(g, \varphi) = 0$ if $\text{Fix}(g) \cap \text{Coinc}(\varphi, p) = \emptyset$, etc. These details are left to the reader. Lefschetz-trace formulas for $J(g, \varphi)$ can be found in 2.1 and 3.5.