§ 1. The coincidence-fixed-point (c.f.p.) index

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§ 1. The coincidence-fixed-point (c.f.p.) index

(1.1) Let $p: E \to B$ denote a euclidean neighborhood retract over B (abbrev. ENR_B), where B, and hence E, is an ENR. Altogether this means that $p: E \to B$ embeds as a neighborhood retract into the projection $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$, for some$m$, n. We refer the reader to [2], §1, for the precise definitions but remark that every smooth submersion and every fibration (with base and total space ENR) qualifies for $p: E \to B$.

We consider continuous maps $g: D_g \to E$, $\varphi: D_\varphi \to B$, where D_g, D_φ are open subsets of E, and $pg = p \mid D_g$ (i.e., g is fibre-preserving). We let $\mathrm{Fix}\,(g) = \{ x \in D_g \mid gx = x \}$ and $\mathrm{Coinc}\,(\varphi, p) = \{ x \in D_\varphi \mid \varphi x = px \}$, and we *assume that* $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ *is compact.* Under these circumstances we shall define an integer $J(g, \varphi) \in \mathbf{Z}$ which is akin to the Hopf fixed-point index. It "counts" the points in $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ in a weighted and homotopy-invariant fashion. It *is* the Hopf index of g resp. φ if B is a single point resp. p is the identity map of B.

(1.2) By definition [2], 1.1 of an ENR_B, we have that E is a fibre-preserving neighborhood retract of some $\mathbf{R}^n \times B$. In fact, for the present purpose we can use any product $Y \times B$, i.e. we'll use mappings $E \xrightarrow{i} V \xrightarrow{r} E$ such that $V \subset Y \times B$ is open, $ri = id$, and i, r are maps over B. In formulas,

(1.3) $$ix = (i'x, px)\,, \quad \text{where } i': E \to Y\,,$$

(1.4) $$pr\,(y, b) = b\,, \quad \text{for } (y, b) \in V\,,$$

(1.5) $$r\,(i'x, px) = x\,, \quad \text{for } x \in E\,.$$

Consider the following sequence of maps

(1.6) $$D_g \cap D_\varphi \xrightarrow{(g,\varphi)} E \times B \xrightarrow{i' \times id} Y \times B \supset V \xrightarrow{r} E\,.$$

Its composite $[g, \varphi]$ is defined in $D_V = (i'g, \varphi)^{-1}\, V$ which is an open subset of $(D_g \cap D_\varphi)$, and hence of E. Thus

(1.7) $$[g,\varphi]: D_V \to E\,, \quad [g, \varphi]\,(x) = r\,(i'gx, \varphi x)\,.$$

If $x \in D_g \cap \mathrm{Coinc}\,(\varphi, p)$ then

$$(i'g, \varphi)\,x = (i'gx, px) = (i'gx, pgx) = igx \in V\,,$$

hence $[g, \varphi]\,x$ is defined and equals $rigx = gx$. It follows that $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p) \subset \mathrm{Fix}\,[g, \varphi] = \{ x \in D_V \mid [g, \varphi]\,x = x \}$. Converseley, x

$= [g, \varphi]\, x$ implies $px = p\,[g, \varphi]\, x = pr\,(i'gx, \varphi x) = \varphi x$, hence $x \in \operatorname{Coinc}(\varphi, p)$, and $gx = [g, \varphi]\, x = x$. Altogether

$$\operatorname{Fix}(g) \cap \operatorname{Coinc}(\varphi, p) = \operatorname{Fix}[g, \varphi]\,. \tag{1.8}$$

In particular, $[g, \varphi]\colon D_V \to E$ has a compact fixed-point set, and we can assign to it its Hopf-index $I\,[g, \varphi] \in \mathbf{Z}$ — for instance as in [1], VII,5.10. Furthermore,

(1.9) Proposition and Definition. *The Hopf-index $I\,[g, \varphi] \in \mathbf{Z}$ depends only on (g, φ), not on the choice of the neighborhood retraction i, r.* We denote this integer by $J\,(g, \varphi)$, and call it the *c.f.p.-index* of (g, φ); thus $J\,(g, \varphi) = I\,[g, \varphi]$.

Proof. Because the range B of the maps φ, p is ENR, these two maps are homotopic in a neighborhood of $\operatorname{Coinc}(\varphi, p)$. In fact (cf. [1], IV,8.6), there is an open neighborhood U of $\operatorname{Coinc}(\varphi, p)$ in D_φ, and a deformation $\vartheta_t\colon U \to B$, $0 \leqq t \leqq 1$, such that

$$\vartheta_0 = p \mid U,\ \vartheta_1 = \varphi \mid U,\ \vartheta_t x = px \text{ for } x \in \operatorname{Coinc}(\varphi, p) \text{ and all } t. \tag{1.10}$$

Consider then two neighborhood retractions

$$E \xrightarrow{i} V \xrightarrow{r} E\,,\ V \subset Y \times B\,;\ ix = (i'x, px)\,,$$
$$E \xrightarrow{j} W \xrightarrow{s} E\,,\ W \subset Z \times B\,;\ jx = (j'x, px)\,,$$

as above, and the corresponding maps $[g, \varphi]_1$, $[g, \varphi]_2$ as defined by 1.6. We have to show $I([g, \varphi]_1) = I([g, \varphi]_2)$. In order to do so we can (cf. [1], VII,5.11) restrict attention to an arbitrary open neighborhood N of $\operatorname{Fix}([g, \varphi]_i) = \operatorname{Fix}(g) \cap \operatorname{Coinc}(\varphi, p)$. And we shall show that $[g, \varphi]_i \mid N$ are homotopic $(i = 1, 2)$ without moving the fixed point set, provided N is sufficiently small. The homotopy is given by the formula

$$\theta_t x = s\big(j'r\,(i'gx, \vartheta_t x)\,,\ \varphi x\big)\,. \tag{1.11}$$

This is defined for (x, t) such that $x \in D_g \cap U$, $v = (i'gx, \vartheta_t x) \in V$, and $w = (j'rv, \varphi x) \in W$; the set of all such (x, t) is an open subset D_θ of $E \times [0, 1]$. If $x \in \operatorname{Fix}(g) \cap \operatorname{Coinc}(\varphi, p)$ then

$$v = (i'gx, \vartheta_t x) = (i'x, px) = ix \in V\,, \text{ and } rv = x\,,$$

hence

$$w = (j'rv, \varphi x) = (j'x, px) = jx \in W\,, \text{ and } \theta_t x = sw = x\,.$$

Therefore, $(\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)) \times [0, 1] \subset D_\theta$, and $(\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)) \subset \mathrm{Fix}\,(\theta_t)$ for all t. It follows that

$$N = \{x \in E \mid (x, t) \in D_\theta \text{ for all } t\}$$

is an open neighborhood of $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ in which the deformation θ is defined (by 1.11).

Suppose now $x \in N$ is a fixed point of θ_t, thus $x = s\,(j'r\,(i'gx, \vartheta_t x), \varphi x)$. Apply p, using 1.4 for s, and get $px = \varphi x =$, hence $\vartheta_t x = px$ by 1.10, hence $r\,(i'gx, \vartheta_t x) = r\,(i'gx, px) = r\,(i'gx, pgx) = rigx = gx$, hence $x = \theta_t x = s\,(j'gx, px) = s\,(j'gx, pgx) = sjgx = gx$; altogether, $x \in \mathrm{Coinc}\,(\varphi, p) \cap \mathrm{Fix}\,(g)$. It follows that the fixed point set $\mathrm{Fix}\,(\theta_t) = \mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ for all t. In particular, $\cup_{t \in [0,1]} \mathrm{Fix}\,(\theta_t)$ is compact, hence (cf. [1].VII,5.15) all θ_t have the same Hopf-index $I\,(\theta_t)$. But $r\,(i'gx, \vartheta_0 x) = r\,(i'gx, px) = r\,(i'gx, pgx) = gx$, hence $\theta_0 x = s\,(j'gx, \varphi x) = [g, \varphi]_2 x$. To calculate θ_1 we first remark that $p\,[g, \varphi]_1 x = \varphi x$, by 1.7 and 1.4; also $r\,(i'gx, \vartheta_1 x) = r\,(i'gx, \varphi x) = [g, \varphi]_1 x$, hence $\theta_1 x = s\,(j'\,[g, \varphi]_1 x, p\,[g, \varphi]_1 x = sj\,[g,\varphi]_1 x = [g, \varphi]_1 x$. □

(1.12) *The product case* $E = F \times B$, $p =$ *projection.* In this case $g\colon D_g \to F \times B$ has the form $g\,(y, b) = (\gamma\,(y, b), b)$ with $\gamma\colon D_g \to F$. The two maps (γ, φ) combine to a map $(\gamma, \varphi)\colon D \to F \times B$, where $D\,(= D_g \cap D_\varphi)$ is an open subset of $F \times B$, and $\mathrm{Fix}\,(\gamma, \varphi) = \mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$. In order to obtain the c.f.p.-index $J\,(g, \varphi)$ one can use $Y = F$ and the neighborhood retraction $i = r =$ identity-map of $Y \times B$. The definition 1.9 then shows that

$$J\,(g, \varphi) = I\,(\gamma, \varphi)\,;$$

i.e. *in the product case the c.f.p.-index of* (g, φ) *is simply the Hopf-index of* $(y, b) \mapsto (\gamma\,(y, b), \varphi\,(y, b))$.

The procedure 1.6-1.9 in the general case, on the other hand, can be considered as a reduction to the product case.

(1.13) *General properties of* $J\,(g, \varphi)$ follow from corresponding properties of the Hopf-index. For instance, $J\,(g, \varphi)$ is additive with respect to topological-sum decompositions of $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(g, \varphi)$, it is invariant under deformations such that $\bigcup_{0 \leq t \leq 1} \mathrm{Fix}\,(g_t) \cap \mathrm{Coinc}\,(\varphi_t, p)$ is compact, it depends only on the germ of (g, φ) around $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p)$ — in particular, $J\,(g, \varphi) = 0$ if $\mathrm{Fix}\,(g) \cap \mathrm{Coinc}\,(\varphi, p) = \varnothing$, etc. These details are left to the reader. Lefschetz-trace formulas for $J\,(g, \varphi)$ can be found in 2.1 and 3.5.