14. Duality

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principally this functor (as it was needed for the universal coefficient theorem in cohomology) that led Eilenberg-Mac Lane in 1943 to the step of introducing categories in general and functors on them, both covariant and contravariant.

The categorical language was soon generally used for homology theory and homological algebra—but one essential element of that language was missing: The notion of adjoint functor. This notion did not actually appear till D. M. Kan's clear introduction in 1958. To be sure, many special examples, usually under the form of a suitable universal property, had been long present. However, the great merit of the notion lies in its generality and systematic availability. In retrospect (see Mac Lane [1976]) it is strange indeed that it took 15 years from the introduction of categories in 1943 to the definition of adjoint functors in 1958. It may indeed be that there was a widespread prejudice against very general notions ("general abstract nonsense") and that the mores of mathematical research were determined more by a sort of positivistic view—all that matters are hard calculations leading to explicit theorems solving known problems. This clearly useful and effective standard—for most mathematical purposes—may have needlessly inhibited the development of appropriate general concepts. This is hard to judge with certainty. I do know that Eilenberg-Mac Lane for a dozen years after their initial publication on category theory considered that category theory was chiefly a language, and that further serious research in the subject was not worth trying. When Daniel Kan, coming from outside the main communities of mathematics, did arrive at the notion of a pair of adjoint functors, his work was warmly greeted by Eilenberg.

This may leave us to wonder if there are other general notions not yet discovered which might be useful for the organization of mathematics.

14. Duality

One general notion, that of categorical duality and its topological application, did not lack for attention. Pontryagin duality for topological groups had long (since about 1930) been a central tool for the algebraic topologists, especially for its use with the coefficient groups of knowledge and cohomology. The alternative possibility of dualities which are axiomatic (because they arise from a dual involution of the undefined terms of an axiom system) could not very well become relevant for topology until the categorical language was available. Possibly the first step in this direction was the proof (about 1940) by Reinhold Baer that the dual of a free group (in effect, the dual taken in the category of all groups) was necessarily a one-element group. This result may even have had some political overtones, since the dual of "free" might then have been labelled "fascist".

In 1948 Mac Lane, during a four-month stay in Zurich, observed that the use of categories would allow the exact formulation of the notion of the dual of a theorem about a category—by reversing both the arrows and the composition in the statement (in presently more fashionable terminology, by taking the original theorem for the opposite category). Mac Lane's first paper on this subject, in the *Proceedings* of the National Academy of Sciences, dealt chiefly with such dualities for the category of groups. This study did not lead very far, because the duals of many true theorems in this category are not true—and one has till this day no real understanding of the class of theorems on groups for which such duality would hold. Mac Lane's second paper [1950] on this topic was concerned more with categorical ideas, especially the introduction of what is essentially the notion of an abelian category (his axioms were too clumsy because he tried to get an exact duality between subobjects and quotient objects; later it became clear that duality "up to isomorphism" suffices). This *should* have even been clear at the time; specifically, the same paper presented the (now familiar) categorical definition of direct product and free product—a definition by diagrams which identifies these products only "up to isomorphisms".

Duality considerations for the category of topological spaces turned out to be much more profitable. The essential observation here is that the covering homotopy theorem (and consequently, the notion of a fiber map) is the dual of the homotopy extension theorem (and the notion of a cofiber map). I have not succeeded in determining who first observed this duality, but it is clear that the team of Eckmann and Hilton most effectively formulated this idea (in their terms, projective and injective homotopy). This they began with three notes in the *Comptes Rendus* in 1958, and continued in a considerable sequence of papers, in particular, the three papers [1962-1963] on group-like structures in general categories. Of these, the first 1958 note considered group structure on the set $\Pi\,(A, B)$ of homotopy classes of maps of the space A into the space B. They proved that an H-space structure on B gave a group structure on $\Pi\,(A, B)$ which is natural in A and dually that a H'-space structure on A yields a group structure on $\Pi\,(A, B)$ which is natural in B. Here too they proved the beautiful easy theorem that for A an H'-space and B an H-space the two group structures on $\Pi\,(A, B)$ agree and are abelian, observing the consequence that higher homotopy groups are abelian. They used systematically the reduced suspension Σ, the loop space construction Ω and the adjunction

$$\Pi(\Sigma A, B) \cong \Pi(A, \Omega B)$$

(though they did not explicitly note that this made Σ left adjoint to Ω). They went on to define higher homotopy groups

$$\Pi_n(A, B) = \Pi(\Sigma^n A, B) = \Pi(A, \Omega^n B)$$

corresponding relative groups and the appropriate long exact sequences. These long exact sequences, which extended Barratt's 1955 "track group sequences" were further codified by D. Puppe and are now the Puppe sequences. Eckmann's report at the 1962 International Congress gives an especially clear formulation of this whole set of ideas (including the notion of spectra).

Our main contention is that the systematic use of cohomology of groups and the resulting categorical ideas inevitably led to the systematic use of duality in algebraic topology. We have not tried here to trace the exact authorship of these ideas—because it is clear that many topologists played a role in this work. John Moore was concerned with Eilenberg-Mac Lane spaces $K(\Pi, n)$—the spaces arising from the cohomology of groups with only one homotopy group Π in dimension n; in the 1954 Cartan seminar he introduced the (quasi-dual) Moore spaces $K'(\Pi, n)$—with only one homology group Π in dimension n. At about that time he and others must have considered the "duals" of the Postnikov decomposition of a map—a notion explicitly formulated in the fourth Eckmann-Hilton note in *Comptes Rendus* (1959). E. H. Brown's work (1962) on the Representation of Cohomology Theories, and George Whitehead on Generalized homology theories (1962), also belong here. These ideas were surely "in the air".

One historical note on these ideas did turn up during the Zurich conference. Given a cohomology theory h^* defined by a spectrum B and given a polyhedron A, there is a spectral sequence E_n^{pq} starting with the ordinary cohomology $E_2^{pq} = H^p(A, h^q(S_0))$ and converging to (the graded module associated to a filtration of) $h^{p+q}(A)$. This spectral sequence is usually called the Atiyah-Hirzebruch spectral sequence, because it first appeared in print for the case when h^* is K-theory in a paper (1961) by these authors. The background, as told me by J. F. Adams, is as follows: On August 4, 1955, George Whitehead has submitted to the *Transactions* of the American Mathematics Society a paper (1956) on the homotopy groups of joins and unions. In modern language, it gave for stable homotopy Π_*^S a spectral sequence $H_* (X, \Pi_*^S Y) \Rightarrow \Pi_*^S (X * Y)$, where $X * Y$ is the join of the spaces X and Y. In discussion with Adams, Whitehead talks about his definition of a generalized homology theory K and said that his paper "should" have proved $H_* (X, K_* (\mathrm{pt})) \Rightarrow K_* (X)$. Later, Atiyah told Adams about his joint work with Hirzebruch on K-theory as a generalized cohomology; he also wondered about its relation to ordinary cohomology. Adams, recalling the words of Whitehead, observed that there was a suitable spectral sequence; Atiyah asked how it was constructed and whether it was published. Adams thus reported that it was constructed in the inevitable way, from an appropriate filtration—but that it had not been published. Atiyah resigned himself to the trouble of writing it up—and so it is now called the Atiyah-Hirzebruch sequence. Given the familiarity at that time with the technique of spectral sequences, it is clear that this sequence was sure to be discovered at about that time—if not by one author, then by another.

15. Cohomology of Algebraic Systems

The cohomology of groups was just the starting point for the study of corresponding cohomology theorems of other sorts of algebraic systems. A few months after the discovery of the cohomology of groups, Hochschild found a corresponding cohomology for algebras. Again, the 2-dimensional cohomology group of an algebra corresponded to an extension problem for algebras, and it soon turned out that the Eilenberg-Mac Lane interpretation of H^3 as obstructions for non-abelian extensions of groups could also be carried over to algebras. Presently Chevalley and Eilenberg formulated a cohomology theory for Lie algebras. It was now amply clear that the idea of cohomology, originally conceived as a measure of the connectivity of spaces, was also relevant as a record of some of the aspects of quite a variety of algebraic systems. The connection with topology remained strong, however. For example, the Eilenberg-Mac Lane spaces $K(\Pi, n)$ were defined topologically, as spaces with Π the only non-vanishing homotopy group- in dimension n; their stable cohomology, however, could be interpreted as the cohomology of the abelian group Π (Mac Lane [1950]). This cohomology—and that of other algebraic systems—can be calculated systematically from a complex which is "generically acyclic" in the sense of Eilenberg-Mac Lane [1951] [1955]. The full meaning of this notion is still mysterious.