10. Transfer

$$E_2^{pq} \cong H^p(Q, H^q(K, M)) \tag{2}$$

converging to the graded group associated with a filtration of the cohomology $E^{p+q}(G, M)$. In (2), the cohomology $H^q(K, M)$ of the subgroup K is suitably interpreted as a Q-module, so that the outside cohomology is defined. The essential portions of such a spectral sequence were discovered by R. Lyndon in his 1946 Harvard thesis, at about the same time that Leray was formulating the general notion of a spectral sequence. Lyndon did use his formulation for computation. Some years later [1953], Hochschild and Serre formulated a spectral sequence like that of (2) in the conventional language, so such a sequence is usually called a Hochschild-Serre spectral sequence. (There are actually several different constructions of such a sequence, and some residual uncertainty as to whether these constructions all yield the same spectral sequence). The essential observation is that computing cohomology or homology in a fiber situation like that of (1) inevitably leads to the spectral sequence technology—whether the fiber situation is group theoretic, as with the exact sequence (1), or a fiber space, as in the case so effectively exploited by Serre in topology.

10. Transfer

The operation of *transfer* was well known in group theory, beginning with Burnside's work on monomial representations. If H is a subgroup of index n in G, the transfer from G to H is a homomorphism.

$$t : G/[G, G] \to H/[H, H] \tag{1}$$

between the factor-commutator groups. To define it, choose representatives $x_1, \ldots, x_n$ of the right cosets of H in G, so that $G = \cup Hx_i$ and write $\rho(x)$ for the representative x_i of the coset Hx. Then t is

$$t(g) = \prod_{i=1}^{n} (x_i g)\, [\rho(x_i g)]^{-1} \tag{2}$$

This map t is independent of the choice of the set of representatives $x_i, \ldots, x_n$.

Since the factor commutator group $G/[G, G]$ in (1) is simply the 1-dimensional homology group $H_1(G, \mathbf{Z})$, the transfer can be regarded as a map in homology.

$$t : H_1(G, \mathbf{Z}) \to H_1(H, \mathbf{Z})$$

In 1953 Eckmann extended this map to apply in all dimensions, both in homology and cohomology. Using the standard homogeneous complexes

$B(G)$ and $B(H)$ for the groups G and H, he defined a cochain transformation t for any G-module A and any cochain f by

$$(tf)(g_0, \ldots, g_p) = \sum_{i-1}^{n} x_j^{-1} f(x_j g_0 (\rho(x_j g_0))^{-1}, \ldots, x_j g_n (\rho(x_j g_n)) - 1)$$

This map, up to chain homology, is again independent of the choice of the representatives x_j, so yields a homomorphism

$$t : H^p(H, A) \to H^p(G, A) .$$

On the other hand, each cochain of G over A automatically restricts to a cochain of H over A; this process defines the *restriction* map

$$r : H^p(G, A) \to H^p(H, A) .$$

Eckmann proved that the composite tr of these maps is the endomorphism given by multiplication by n in $H^p(G, A)$: He made a variety of applications. The notion of transfer was also used by Artin and Tate (see below) in class field theory.

The discovery of the homology of a group had the feature that it exhibited a "non-obvious" construction on groups; in much the same way, the discovery of transfer produced a non-obvious homomorphism between cohomology groups. Thus it is that recently Kahn and Priddy have been able to construct the transfer homeomorphism for the generalized cohomology of an n-fold covering $\Pi : E \to B$. This transfer applies to the cohomology with coefficients in any strict Ω-spectrum; when applied to the Eilenberg-Mac Lane spectrum $K(\Pi, n)$, the generalized cohomology is ordinary cohomology and the transfer agrees with the classical one. Using this transfer, they prove a conjecture of Mahowald and Whitehead about a "canonical map" of the n-fold suspension $\Sigma^n \mathbf{R}P^{n-1}$ of the real projective $n - 1$ space into the n sphere. This map λ is the adjoint of the map

$$\mathbf{R}P^{n-1} \to O_n \to \Omega^n S^n .$$

Here the first arrow takes a line through the origin in $\mathbf{R}^n$ into the reflection in the plane perpendicular to that line; while the second arrow represents each element of O^n as a map of $(\mathbf{R}_n \cup \infty, \infty)$ into itself, and hence as an element of the n^{th} loop space of S^n.

The result of Kan and Priddy is that λ is an epimorphism of 2-primary components in stable homotopy.