5. Application to schemes

At this point, k is a field (the quotient field of $A_0/\mathfrak{P}$) and R is a graded algebra over the field k, so all assumptions of theorem B are fulfilled. Moreover let ε the composition of the natural maps

$$A \to A' \to A'' \to R .$$

In degree 0, ε_0 is nothing else than the natural map from A_0 into k with kernel $\mathfrak{P}$. Since φ has the same kernel $\mathfrak{P}$, it factors through ε_0, making K an algebraically closed extension of k.

We quote now theorem B. There exists a k-linear ring homomorphism $f : R \to K$ such that $f(R^+) \neq 0$. The composite map $\Psi = f\varepsilon$ has all the required properties.

5. Application to schemes

We keep the notation of theorem D. Recall that the spectrum $S = \mathbf{Spec}\,(A_0)$ of A_0 is the set of all prime ideals in A_0; the projective spectrum $X = \mathbf{Proj}\,(A)$ of A is the set of all *graded* prime ideals in A, which do not contain the ideal $A^+ = \bigoplus_{d \geqq 1} A_d$. We have a natural map $\pi : X \to S$ associating to every graded prime ideal $\mathfrak{P}$ in A the prime ideal $\mathfrak{P} \cap A_0$ in A_0.

Moreover S and X are endowed with their respective Zariski topologies. A set F in S (resp. X) is closed if and only if there exists an ideal $\mathfrak{A}$ in A_0 (resp. A) such that F is the set of ideals $\mathfrak{P}$ of S (resp. X) containing $\mathfrak{A}$. It is obvious that π is continuous.

The following theorem is Grothendieck's version of the elimination theorem. Using his language, it is the main step in the proof that $X = \mathbf{Proj}\,(A)$ is a proper scheme over $S = \mathbf{Spec}\,(A_0)$.

Theorem E. *The map $\pi : X \to S$ is closed, that is the image of a closed set is closed.*

Let $F \subset X$ be closed and let $\mathfrak{A}$ be an ideal in A such that F consists of the graded prime ideals $\mathfrak{P}$ of X containing $\mathfrak{A}$. Replacing if necessary $\mathfrak{A}$ by the ideal generated by the homogeneous components of its elements, we may and shall assume that $\mathfrak{A}$ is a graded ideal. Let $\mathfrak{B}$ be the set of elements a in A_0 such that $a \,.\, A_d \subset \mathfrak{A}$ for large d, and let G be the set of prime ideals in A_0 containing $\mathfrak{B}$. It is obvious that π maps F into G.

Let $\mathfrak{P}_0$ be a prime ideal in G, hence $\mathfrak{P}_0 \supset \mathfrak{A}_0$ (where $\mathfrak{A}_0 = \mathfrak{A} \cap A_0$). Denote by k the quotient field of $A_0/\mathfrak{P}_0$ and by K an algebraically closed overfield of k. Let φ be the natural composite map $A_0/\mathfrak{A}_0 \to A_0/\mathfrak{P}_0 \to k \to K$. We are now in a position to apply theorem D to the graded ring $A/\mathfrak{A}$, and we get a ring homomorphism $\Psi : A/\mathfrak{A} \to K$ extending φ and such that $\Psi((A^+ + \mathfrak{A})/\mathfrak{A}) \neq 0$. Let $\mathfrak{P}_d$ (for $d \geqslant 1$) be the set of elements a in A_d such that $\Psi(a + \mathfrak{A}) = 0$. Then $\mathfrak{P} = \bigoplus_{d \geq 0} \mathfrak{P}_d$ is a graded prime ideal in A containing $\mathfrak{A}$ with $\mathfrak{P} \not\supset A^+$ and $\mathfrak{P} \cap A_0 = \mathfrak{P}_0$. That is, $\mathfrak{P}$ belongs to F and π maps $\mathfrak{P}$ onto $\mathfrak{P}_0$.

(Reçu le 18 mars 1978)

P. Cartier

Institut des Hautes Etudes Scientifiques
F-91440 — Bures-sur-Yvette

J. Tate

Harvard University
Cambridge, Mass. 02138