3. Elimination theory

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$I_0 = (0)$ and $\mathfrak{I}_0 = \mathfrak{I}$ and define inductively r_d, I_d and $\mathfrak{I}_d$ as follows. For $d \geqslant 0$, let r_{d+1} be equal to the maximum of the dimensions of $I \cap R_{d+1}$ for I running over $\mathfrak{I}_d$, let I_{d+1} be any ideal in $\mathfrak{I}_d$ such that $\dim (I_{d+1} \cap R_{d+1}) = r_{d+1}$ and let $\mathfrak{I}_{d+1}$ be the set of ideals I in $\mathfrak{I}_d$ such that $I \cap R_{d+1} = I_{d+1} \cap R_{d+1}$. Then the ideal $\bigoplus_{d \geq 1} (I_d \cap R_d)$ is a maximal element in $\mathfrak{I}$, as it is easily checked.

3. Elimination theory

The main theorem of elimination theory may be formulated as follows. Let $P_1, \ldots, P_r$ be polynomials in $k[X_0, X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ with P_j homogeneous of degree d_j in the variables $X_0, X_1, \ldots, X_n$ alone, i.e. of the form

$$P_j = \sum_{\alpha_0 + \ldots + \alpha_n = d_j} X_0^{\alpha_0} X_1^{\alpha_1} \ldots X_n^{\alpha_n} f_{\alpha,j}(Y_1, \ldots, Y_m)$$

where the $f_{\alpha,j}$'s are polynomials in $k[Y_1, \ldots, Y_m]$.

Denote by J the ideal in $k[X_0, X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ generated by $P_1, \ldots, P_r$ and by $\mathfrak{A}$ the ideal of polynomials f in $k[Y_1, \ldots, Y_m]$ with the following property (the so-called Hurwitz' Trägheitsformen):

(E) There exists an integer $N \geqslant 1$ such that $f X_0^N, f X_1^N, \ldots, f X_n^N$ all belong to J.

As usual we denote by $\mathbf{P}^n(K)$ the n-dimensional projective space over K.

Theorem C. *Let V be the subset of $\mathbf{P}^n(K) \times K^m$ consisting of the pairs (x, y) with $x = (x_0 : x_1 : \ldots : x_n)$ and $y = (y_1, \ldots, y_m)$ such that $P_j(x_0, x_1, \ldots, x_n; y_1, \ldots, y_m) = 0$ for $1 \leqslant j \leqslant r$. Let W be the subset of K^m consisting of the vectors y such that $Q(y) = 0$ for every Q in $\mathfrak{A}$. Then the projection of $V \subset \mathbf{P}^n(K) \times K^m$ onto the second factor K^m is equal to W.*

To reformulate theorem C, let us consider the ring

$$B = k[X_0, X_1, \ldots, X_n; Y_1, \ldots, Y_m]$$

together with its subring $B_0 = k[Y_1, \ldots, Y_m]$. Denote by B_d the B_0-module generated in B by the monomials of degree d in $X_0, X_1, \ldots, X_n$. Then $B = \bigoplus_{d \geq 0} B_d$ is a graded ring with J a graded ideal. Define the *graded ring* $A = B/J$ with $A_d = B_d/(B_d \cap J)$. We have the following properties:

(i) *As a ring,* A *is generated by* $A_0 \cup A_1$.

(ii) *For any nonnegative integer* d, A_d *is a finitely generated module over* A_0.

Furthermore, let $\mathfrak{S}$ be the ideal in A_0 consisting of all a's such that $aA_d = 0$ for all sufficiently large d's, i.e. the union of the annihilators of the A_0-modules $A_0, A_1, A_2, \ldots$.

THEOREM D. *Let* $A = \bigoplus_{d \geq 0} A_d$ *be a graded commutative ring obeying hypotheses* (i) *and* (ii) *above. Let* K *be an algebraically closed field and* $\varphi : A_0 \to K$ *be a ring homomorphism. In order that* φ *extend to a ring homomorphism* $\Psi : A \to K$ *which does not annihilate the ideal* $A^+ = \bigoplus_{d \geq 1} A_d$ *in* A, *it is necessary and sufficient that* φ *annihilate the ideal* $\mathfrak{S}$ *defined above.*

We leave to the reader the simple proof of the necessity in theorem D as well as the derivation of theorem C from theorem D.

4. PROOF OF THEOREM D

Let $\mathfrak{P}$ be the kernel of φ, a prime ideal in A_0. Assume $\mathfrak{S} \subset \mathfrak{P}$. We subject the ring A to a number of transformations. At each step, the properties (i) and (ii) enunciated before the statement of theorem D will be preserved, as well as property $A_d \neq 0$ for every $d \geqslant 0$. We shall mention what has been achieved after each step.

a) Factor A through the following graded ideal J: an element a in A belongs to J if and only if there exists an element s in A_0 such $s \notin \mathfrak{P}$ and $sa = 0$. For every $d \geqslant 0$, the annihilator $\mathfrak{S}_d$ of the A_0-module A_d is contained in $\mathfrak{S}$ hence in $\mathfrak{P}$ and this implies $J \cap A_d \neq A_d$. Put $A' = A/J$, $\mathfrak{P}' = (\mathfrak{P} + J)/J$ and $\Sigma = A_0' - \mathfrak{P}'$. Then *any element in* Σ *is regular in* A'.

b) Enlarge A' by replacing it by the subring A'' of the total quotient ring of A' consisting of the fractions with denominators in Σ. Let A_d'' be the set of fractions with numerator in A_d' and denominator in Σ; then $A'' = \bigoplus_{d \geq 0} A_d''$. Then A_0'' *is a local ring with maximal ideal* $\mathfrak{P}'' = \mathfrak{P}' \,.\, A_0'$.

c) Factor A'' through the graded ideal $\mathfrak{P}'' \,.\, A''$. Since A_d'' is a finitely generated module over the local ring A_0'', one gets $A_d'' \neq \mathfrak{P}'' A_d''$ by Nakayama's lemma. Put $k = A_0'' \backslash \mathfrak{P}''$, and $R = A''/\mathfrak{P}'' A''$.