

# 3. Tabulation of the Degrees

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### 3. TABULATION OF THE DEGREES

Theorem 3.8 can be used to compute the degrees of the basic homogeneous invariants of  $G$ , in case  $G$  is an irreducible reflection group acting on  $R^n$ . This has been done in [7], and we tabulate these degrees below

Group	$d_1, \dots, d_n$
$A_n (n \geq 1)$	$2, \dots, n + 1$
$B_n (n \geq 2)$	$2, 4, \dots, 2n$
$D_n (n \geq 4)$	$2, 4, \dots, n, \dots, 2n - 4, 2n - 2$
$H_2^n (n \geq 5)$	$2, n$
$E_6$	$2, 5, 6, 8, 9, 12$
$E_7$	$2, 6, 8, 10, 12, 14, 18$
$E_8$	$2, 8, 12, 14, 18, 20, 24, 30$
$F_4$	$2, 6, 8, 12$
$I_3$	$2, 6, 10$
$I_4$	$2, 12, 20, 30$

We observe that in each case,  $d_1 = 2$ . This can be seen as follows. Suppose that there existed a homogeneous invariant  $I(x)$  of degree 1. Since  $I(\sigma x) = I(x)$  whenever  $\sigma \in G$ , the hyperplane  $\{x \mid I(x) = 0\}$  would be a proper invariant subspace of  $G$ , contradicting that the latter is irreducible. Hence there are no homogeneous invariants of degree 1 and  $d_1 \geq 2$ . On the other hand,  $\sum_{i=1}^n x_i^2$  is invariant under  $G$  as  $G$  is orthogonal. It follows

that  $d_1 = 2$ , with corresponding invariant  $I_1 = \sum_{i=1}^n x_i^2$ .

In applying Theorem 3.8, we must find the roots of the characteristic equation (3.23). In some cases, this is a rather tedious computation. For the groups  $A_n, B_n, D_n, H_2^n$  we can exhibit a basis of homogeneous invariants without the use of Theorem 3.8. We require

**THEOREM 3.13.** *Let  $G$  be a finite reflection group acting on the  $n$ -dimensional vector space  $V$  over a given field  $k$ . Let  $P_1, \dots, P_n$  be homogeneous*

invariants of  $G$  of respective degrees  $k_1, \dots, k_n$ .  $P_1, \dots, P_n$  form a basis for the invariants of  $G \Leftrightarrow k_1 \dots k_n = |G|$  and

$$\Delta = \frac{\partial (P_1, \dots, P_n)}{\partial (x_1, \dots, x_n)} \neq 0.$$

*Proof.* By relabeling indices, we may assume  $k_1 \leq \dots \leq k_n$ . The  $\Rightarrow$  part of the theorem is contained in Theorems 1.2, 2.2, 2.3. Conversely, let  $k_1 \dots k_n = |G|$  and  $\Delta \neq 0$ . Thus  $P_1, \dots, P_n$  are algebraically independent. Let  $I_1, \dots, I_n$  be basic homogeneous invariants of respective degrees  $d_1, \dots, d_n$ . Suppose  $k_i = d_i, 1 \leq i \leq i_0$ , but  $k_{i_0+1} < d_{i_0+1}$ . Then  $P_1, \dots, P_{i_0+1}$  are polynomials in  $I_1, \dots, I_{i_0}$ , implying that  $P_1, \dots, P_n$  are algebraically dependent, a contradiction. Hence  $k_i \geq d_i, 1 \leq i \leq n$ . Since

$$\prod_{i=1}^n d_i = \prod_{i=1}^n k_i = |G|, \text{ we must have } k_i = d_i, 1 \leq i \leq n.$$

Let  $\delta_m = \dim \mathcal{J}_m, 0 \leq m < \infty, \mathcal{J}_m$  being the space of homogeneous invariants of degree  $m$ . Then  $\delta_m =$  number of non-negative integral solutions to  $j_1 d_1 + \dots + j_n d_n = m$ . This number also equals the number of monomials  $P_1^{j_1} \dots P_n^{j_n}$  which are of degree  $m$ . The algebraic independence of  $P_1, \dots, P_n$  implies that these  $\delta_m$  monomials are linearly independent over  $k$ . Thus  $\mathcal{J}_m$  is spanned by these monomials for  $0 \leq m < \infty$ . We have shown that every homogeneous invariant is a polynomial in  $P_1, \dots, P_n$ , so that the  $P_i$ 's form a basis for the invariants of  $G$ .

We now obtain an explicit basis for the invariants of  $A_n, B_n, D_n, H_2^n$ .  $A_n$ : This group consists of the  $(n+1)!$  permutations  $x'_i = x_{\sigma(i)}, 1 \leq i \leq n+1$ , restricted to the subspace  $V = \{x \mid x_1 + \dots + x_{n+1} = 0\}$ .

We choose  $x_1, \dots, x_n$  as coordinates on  $V$ . Let  $P_i = \sum_{j=1}^{n+1} x_j^{i+1}, 1 \leq i \leq n$ ,

where  $x_{n+1} = -(x_1 + \dots + x_n)$ .  $P_i$  is a homogeneous invariant of degree  $i+1$ . We have  $2 \cdot \dots \cdot (n+1) = (n+1)! = |A_n|$ .

We show that  $\Delta \neq 0$ . Now

$$\frac{\partial P_i}{\partial x_j} = (i+1)x_j^i - (i+1)x_{n+1}^i, 1 \leq i, j \leq n.$$

Hence  $\Delta = (n+1)! D$  where  $D$  is the  $n \times n$  determinant whose  $(ij)$ -th entry  $= x_j^i - x_{n+1}^i$ . To evaluate  $D$ , we introduce the Vandermonde determinant

$$\begin{vmatrix} 1 & \dots & \dots & \dots & 1 \\ x_1 & \dots & \dots & \dots & x_{n+1} \\ x_1^n & \dots & \dots & \dots & x_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j \leq n+1} (x_j - x_i)$$

Subtracting the  $(n+1)$ -th column from the first  $n$  columns, the above determinant is readily seen to equal  $(-1)^n D$ . Thus

$$(3.25) \quad \Delta = (-1)^{n+2} (n+1)! \prod_{1 \leq i < j \leq n+1} (x_j - x_i) = (n+1)! \prod_{1 \leq j \leq n} (x_j - x_i) \cdot \prod_{i=1}^n (x_i + s)$$

where  $s = x_1 + \dots + x_n$ . (3.25) shows that  $\Delta \neq 0$ . We conclude that  $d_1 = 2, \dots, d_n = n + 1$ .

$B_n$ : Let  $P_i = \sum_{j=1}^n x_j^{2i}, 1 \leq i \leq n$ .  $P_i$  is a homogeneous invariant of degree  $2i$ . We have  $2 \cdot \dots \cdot 2n = 2^n n! = |B_n|$ . A computation shows that  $\Delta = 2^n n! \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$ . It follows that  $d_1 = 2, \dots, d_n = 2n$ .

$D_n$ : Let  $P_1 = x_1 \dots x_n, P_i = \sum_{j=1}^n x_j^{2(i-1)}, 2 \leq i \leq n$ .  $P_1$  is a homogeneous invariant of degree  $n$ ;  $P_i, 2 \leq i \leq n$ , is a homogeneous invariant of degree  $2(i-1)$ . The product of the degrees =  $n \cdot 2 \cdot 4 \cdot \dots \cdot (2n-2) = 2^{n-1} n! = |D_n|$ .

$$(3.26) \quad \Delta = \begin{vmatrix} \frac{P_1}{x_1} & \dots & \frac{P_1}{x_n} \\ 2x_1 & \dots & 2x_n \\ \cdot & \dots & \cdot \\ 2(n-1)x_1^{2n-3} & \dots & 2(n-1)x_n^{2n-3} \end{vmatrix} = 2^{n-1} (n-1)! \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$$

It follows that  $d_i, \dots, d_n$  are identical with the numbers  $2, 4, \dots, n, \dots, 2n - 4, 2n - 2$ .

$H_2^n$ : Let  $z$  be the complex coordinate  $x_1 + i x_2$ .  $H_2^n$  may be described as the group generated by the transformation  $z \rightarrow \bar{z}, z \rightarrow \zeta z$ , where  $\zeta = e^{\frac{2\pi i}{n}}$ . Let  $P_1 = x_1^2 + x_2^2, P_2 = Re z^n$ .  $P_1, P_2$  are homogeneous invariants of respective degrees  $2, n$ . The product of these degrees =  $2n = |H_2^n|$ . A computation yields

$$\frac{\partial (P_1, P_2)}{\partial (x_1, x_2)} = -2n Im z^n \neq 0.$$

It follows that  $d_1 = 2, d_2 = n$ .