CHAPTER III THE DEGREES OF THE BASIC INVARIANTS

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If $V_i \not\subset \pi$, then $V_i + \pi = V$ and we conclude from (2.22) that $\dim V_i = \dim (V_i \cap \pi) + 1$. I.e. $V_i \cap \pi$ is a hyperplane in V_i and $\sigma|_{V_i}$ a reflection on V_i. Choose $u \in V_i - \pi$ so that u is an eigenvector of σ. u is a multiple of the root v, so that $v \in V_i$. Thus $\sigma|_{V_i}$ is a reflection of V_i if $v \in V_i$, and the identity if $v \notin V_i$. Furthermore, each root v is in some V_i, $r + 1 \leqslant i \leqslant s$, otherwise the corresponding reflection σ would have been the identity.

Let $\tilde{G}_i$ = subgroup generated by those reflections whose roots are in V_i, $1 \leqslant i \leqslant s$. It is readily checked that $G = \tilde{G}_1 \times \ldots \times \tilde{G}_s$, $G_i = \tilde{G}_i|_{V_i}$. If $\sigma \in \tilde{G}_i$ and $\sigma|_{V_i}$ = identity then σ = identity. The mapping $\sigma \to \sigma|_{V_i}$ is thus an isomorphism from $\tilde{G}_i$ onto G_i.

THEOREM 2.8. *Let G be a finite reflection group acting on V and decompose V as in Theorem 2.7. Every polynomial invariant under G is a polynomial in the invariant polynomials of $G_1, \ldots, G_s$.*

Proof. For each $v \in V$, write $v = v_1 + \ldots + v_s$, $v_i \in V_i$. By Theorem 2.7, for each $\sigma \in G$, we may write $\sigma v = \sigma_1 v_1 + \ldots + \sigma_s v_s$, $\sigma_i \in G_i$. For any polynomial function $p(v)$ on V, we have $p(v) = \sum_{i=1}^{N} p_{i1}(v_1) \ldots p_{is}(v_s)$ where $p_{ij}(v_j)$ is a polynomial function on V_j. If $p(v)$ is invariant under G, then

$$(2.23) \qquad p(v) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} p(\sigma v) = \sum_{i=1}^{N} I_{i1}(v_1) \ldots I_{is}(v_s)$$

where

$$(2.24) \qquad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \varepsilon G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of G_j.

CHAPTER III

THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case G is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where k is the real field and has the advantage of providing an effective method for computing the degrees. The second method (Theorem 3.14) is valid for an arbitrary field of characteristic 0, but is less effective than the first in the real case.

We first prove that the degrees of the basic invariants are independent of any particular basis.

THEOREM 3.1. *Let* G *a finite reflection group acting on the* n*-dimensional vector space* V. *Let* $I_1, \ldots, I_n$ *be homogeneous polynomials of respective degrees* $d_1 \leqslant \ldots \leqslant d_n$ *forming a basis for the invariants of* G. $d_1, \ldots, d_n$ *are independent of the chosen basis* $I_1, \ldots, I_n$.

Proof. Let $J_1, \ldots, J_n$ be another set of homogeneous invariants forming a basis for the invariants of G. Let $d_1' \leqslant \ldots \leqslant d_n'$ be the respective degrees of $J_1, \ldots, J_n$. We must show that $d_i' = d_i$, $1 \leqslant i \leqslant n$. If not, then let i_0 be the smallest i such that $d_{i_0}' \neq d_{i_0}$, say $d_{i_0}' < d_{i_0}$. Each J_i is a polynomial in those I_i's whose degree $\leqslant \deg J_i$. It follows that for $1 \leqslant i \leqslant i_0$, $J_i = P_i(I_1, \ldots, I_{i_0 - 1})$, $P_i(y_1, \ldots, y_{i_0 - 1})$ being a polynomial in $y_1, \ldots, y_{i_0 - 1}$. Hence $J_1, \ldots, J_{i_0}$ are algebraically dependent over k ([22], Vol. 1, p. 181), contradicting that $J_1, \ldots, J_n$ are algebraically independent over k (Theorem 1.2). Thus $d_i' = d_i$, $1 \leqslant i \leqslant n$.

Theorem 3.1. shows that the numbers $d_1, \ldots, d_n$ are determined by G. We shall give an effective method for the computation of the d_i's in case the underlying field k is real. We first digress to discuss the classification of the finite real reflection groups.

1. THE CLASSIFICATION OF THE FINITE REAL REFLECTION GROUPS

These groups have been classified by Coxeter [6]. We give here a brief description of the theory, as we require it for the computation of the d_i's.

We first observe that we may assume G to be orthogonal.

THEOREM 3.2. *Let* G *be a finite group acting on the* n*-dimensional Euclidean space* R^n. *There exists a non-singular transformation* τ *on* R^n *such that the group* $\tau^{-1} G \tau$ *consists of orthogonal transformations.*

Proof. Let $P(x) = \sum_{\sigma \varepsilon G} (\sigma x, \sigma x)$ where $x = (x_1, \ldots, x_n)$ and (x, y) is the inner product of x and y. For $x \neq 0$, each $(\sigma x, \sigma x) > 0$ so that $P(x) > 0$. Furthermore for $\sigma_1 \in G, P(\sigma_1 x) = \sum_{\sigma \varepsilon G} (\sigma \sigma_1 x, \sigma \sigma_1 x) = \sum_{\sigma \varepsilon G} (\sigma x, \sigma x) = P(x)$. Thus $P(x)$ is a positive definite quadratic form invariant under G. Choose $x = \tau y$ so that $P(\tau y) = (y, y)$. We have $(\tau^{-1}\sigma\tau y, \tau^{-1}\sigma\tau y) = P(\sigma\tau y) = P(\tau y) = (y, y)$, $\sigma \in G$, so that the transformations $\tau^{-1}\sigma\tau$ are orthogonal.

Thus all transformations of G become orthogonal after a suitable linear change of variables. We assume from now on that G is orthogonal. If G is a finite reflection group, this condition is equivalent to demanding that all reflections of G are orthogonal. I.e. for any reflection σ, σ fixes all vectors in the r.h. π and $\sigma(v) = -v$, iff v is perpendicular to π. The two unit vectors perpendicular to π are called roots of G. The set of all roots is called the root system of G.

DEFINITION 3.1. Let F be a region of R^n, G a finite group acting on R^n. F is a fundamental region for G iff:

i) $\sigma_1 F \cap \sigma_2 F = \Phi$ whenever $\sigma_1 \neq \sigma_2$,

ii) $R^n = \bigcup_{\sigma \varepsilon G} \sigma \bar{F}$, $\bar{F}$ being the closure of F.

We remark that it suffices to know i) for $\sigma_1 = e$, the identity of G. For $\sigma_1 F \cap \sigma_2 F = \Phi$ iff $\sigma_1^{-1}(\sigma_1 F \cap \sigma_2 F) = F \cap \sigma_1^{-1}\sigma_2 F = \Phi$. If F is a fundamental region, then so is σF, $\sigma \in G$. The group G permutes these fundamental regions and acts transitively on them.

THEOREM 3.3. *Let G be a finite reflection group acting on R^n. Assume that the roots of G span R^n (G is then called a Coxeter group). The complement of the union of the r.h.'s of G consist of $|G|$ fundamental regions called the chambers of G. G permutes these chambers and acts transitively on them. Each chamber F is bounded by n r.h.'s called the walls of F. Let $r_1, \ldots, r_n$ be the n roots perpendicular to the n walls $W_1, \ldots, W_n$ and pointing into F, and let R_i be the reflection in W_i. The r_i's are linearly independent and $r_i \cdot r_j = -\cos \pi/p_{ij}$, $p_{ii} = 1$ and p_{ij} being an integer $\geqslant 2$ if $i \neq j$. The R_i's generate G.*

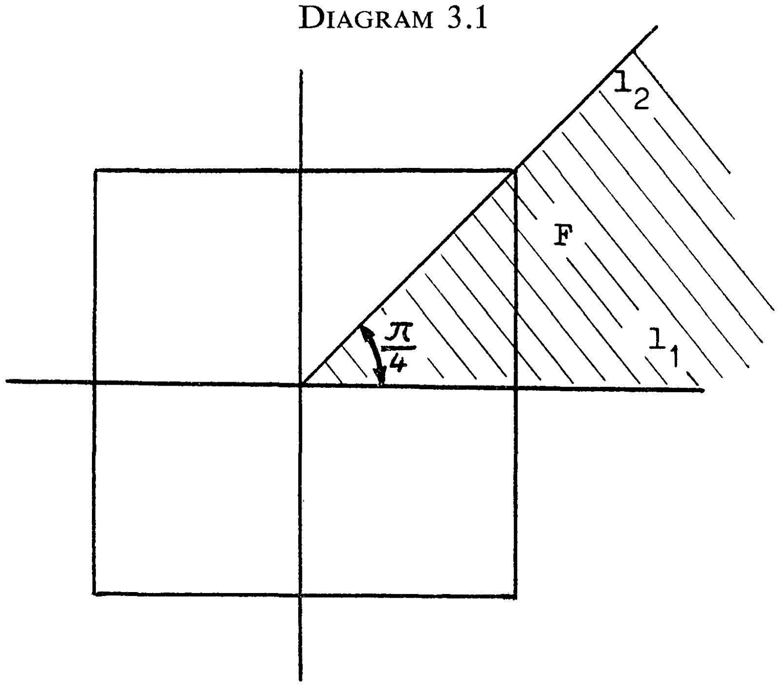
We have $F = \{x \mid x \cdot r_i > 0, 1 \leqslant i \leqslant n\}$. F may also be described as follows. Choose $\{r_1', \ldots, r_n'\}$ to be the dual basis to $\{r_1, \ldots, r_n\}$; i.e. $(r_i, r_j') = \delta_{ij}$. For any x, $x = \sum_{i=1}^{n} (x \cdot r_i) r_i'$. Thus

$$F = \{x \mid x = \sum_{i=1}^{n} \lambda_i r_i', \lambda_i > 0 \text{ for } 1 \leqslant i \leqslant n\}.$$

F is thus a wedge with n walls, the vectors r_i' lying along its edges. The angle between the walls W_i, W_j $(i \neq j)$ is readily seen to be π/p_{ij}. We refer to $\{r_1, ..., r_n\}$ as a fundamental system of roots and to $R_1, ..., R_n$ as a fundamental system of reflections.

As a simple illustration of the above concepts, we choose G to be the group of symmetries of a regular n-gon p_n. G is then called the dihedral group of order $2n$ and we denote it by H_2^n. Assume that the center of the polygon is at the origin. We choose in this case two rays l_1, l_2 emanating from the origin making an angle π/n, one of the rays passing through a vertex of p_n, the other through a mid-point of a side of p_n (see the diagram where $n = 4$). F is the wedge with sides l_1, l_2. The reflections in l_1, l_2 generate H_2^n.

DIAGRAM 3.1



For any Coxeter group G acting on R^n, we introduce the associated Coxeter graph $\mathscr{G}$ as follows. Let $\mathscr{G}$ consist of n points, called the nodes and label these as $1, ..., n$. We set up the $1-1$ correspondence $i \leftrightarrow r_i$, $r_1, ..., r_n$ being the fundamental root system of Theorem 3.3. The i-th and j-th node $(i \neq j)$ are joined by a branch iff $(r_i, r_j) \neq 0$. If this be the case then $p_{ij} \geqslant 3$; we mark the branch joining i to j by p_{ij} whenever $p_{ij} > 3$, and omit a mark if $p_{ij} = 3$. Eg. the graph associated with H_2^n is $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$ for $n = 3$ and $\circ\overset{n}{\text{——}}\circ$ for $n \geqslant 4$.

The motivation for the rather artificial looking definition of $\mathscr{G}$ stems from the following facts.

THEOREM 3.4. *Let G be a Coxeter group acting on R^n. G is irreducible iff its corresponding graph is connected.*

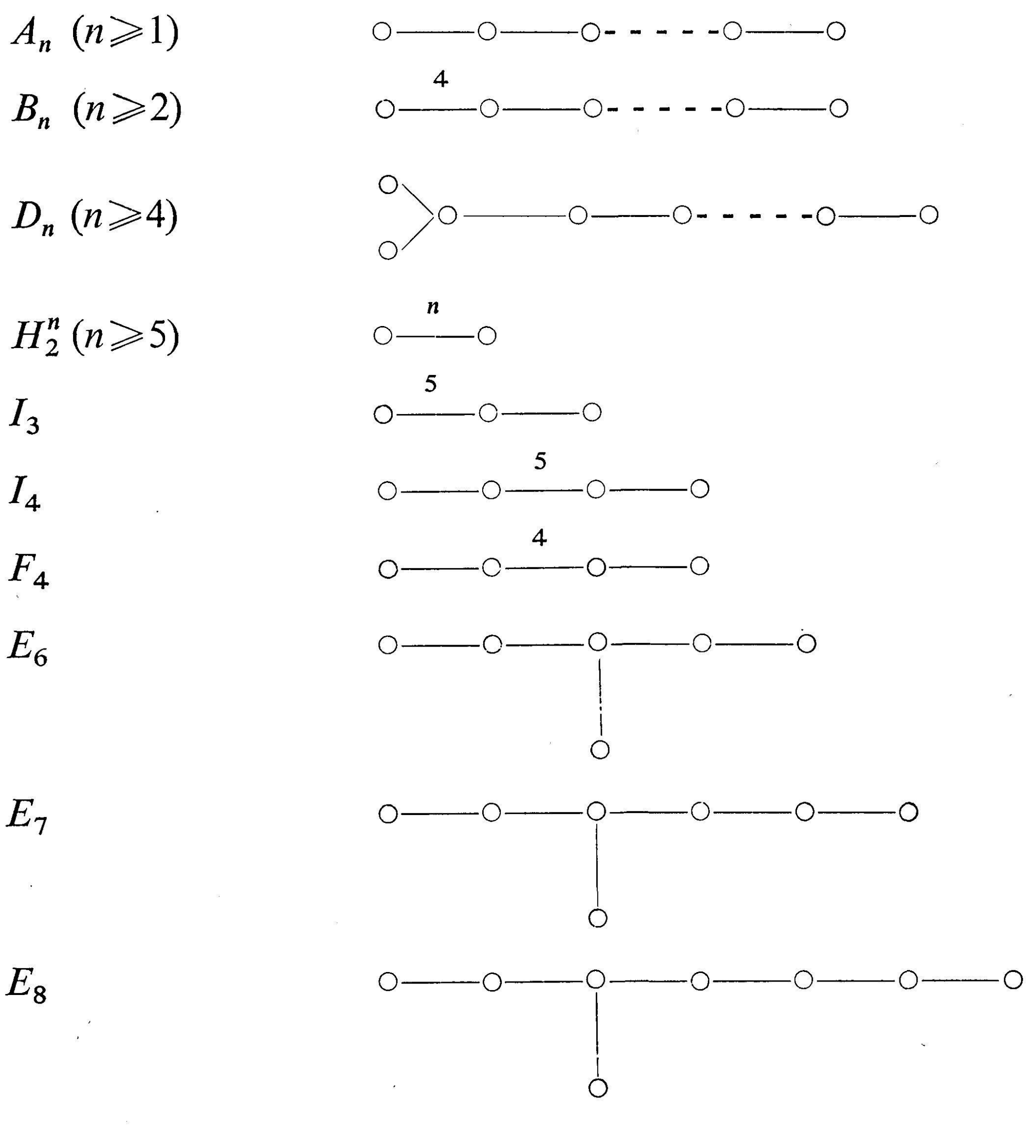
Proof. If the graph of G has more than one component, then the root system $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2$ where $\mathscr{R}_1, \mathscr{R}_2$ are disjoint and non-empty, the roots in $\mathcal{R}_1$ being perpendicular to those in $\mathcal{R}_2$. Let V be the span of the roots in $\mathcal{R}_1$. If σ is a reflection corresponding to a root in $\mathcal{R}_1$, then $\sigma\,|_V$ is a reflection of V. If σ is a reflection corresponding to a root in $\mathcal{R}_2$, then $\sigma\,|_V =$ identity. Since the reflections generate G, V is a proper invariant subspace.

Conversely, let V be a proper invariant subspace of G. Then so is the orthogonal complement $V^{\perp}$. The proof of Theorem 2.7 shows that every root is either in Vor $V^{\perp}$. Since the roots span R^n, there are roots both in V and $V^{\perp}$. Since the roots in $\mathcal{R} \cap V$ are perpendicular to those of $\mathcal{R} \cap V^{\perp}$, the graph of G consists of at least two components.

Coxeter has found all graphs corresponding to the irreducible Coxeter groups. We have the following classification.

THEOREM 3.5. *Let* $\mathcal{G}$ *be a connected Coxeter graph. The following list exhausts the possibilities for* $\mathcal{G}$.

DIAGRAM 3.2



In each case the subscript denotes the number of nodes. The above list yields all irreducible Coxeter groups up to conjugacy. I.e. two irreducible groups which are conjugate subgroups of the orthogonal group have the same graph and conversely.

We give a brief description of the groups listed above.

A_n. Let S_{n+1} be the symmetric group of linear transformations $x_i' = x_{\sigma(i)}$, $1 \leqslant i \leqslant n+1$, $\sigma(i)$ being any permutation of $1, ..., n+1$. Let V $\{x \mid x_1 + ... + x_{n+1} = 0\}$ and $A_n = S_{n+1}|_V$. A_n is the group of symmetries of the regular n-simplex whose vertices are the permutations of $(-1, ..., -1, n)$.

B_n is the group of symmetries of the n cube with vertices $(\pm 1, ..., \pm 1)$. It consists of the $2^n n!$ linear transformations $x_i' = \pm x_{\sigma(i)}$, $1 \leqslant i \leqslant n$, the $\pm$ signs being chosen independently and $\sigma(i)$ an arbitrary permutation of $1, ..., n$.

D_n consists of the $2^{n-1} n!$ linear transformations $x_i' = \pm x_{\sigma(i)}$, $1 \leqslant i \leqslant n$, where $\sigma(i)$ is any permutation of $1, ..., n$ and the number of $-$ signs is even. It is readily checked that D_n is a subgroup of index 2 in B_n.

H_2^n is the dihedral group of $2n$ symmetries of the regular n-gon.

I_3 is the icosahedral group, i.e. the group of symmetries of the icosahedron.

I_4, F_4 are the groups of symmetries of certain 4-dimensional regular polytopes described in ([5], p. 156)

E_6, E_7, E_8 are the groups of symmetries of certain polytopes in R^6, R^7, R^8 known as Gosset's figures and described in ([5], p. 202)

An inspection of diagram 3.2 reveals that the graphs are of two types, those consisting of one chain and those consisting of three chains joined at a node. We refer to these graphs and their associated groups as being of types I and II. It can be shown that the groups of type I are precisely those which are the groups of symmetries of the regular polytopes ([5], p. 199).

The following theorem gives a complete description of all finite reflection groups acting on R^n.

THEOREM 3.6. *Let G be a finite reflection group acting on R^n. R^n is a direct sum of mutually orthogonal subspaces $V_0, V_1, ..., V_k$ with the following properties.*

1) *Let $G_i = G/_{V_i} =$ the restrictions of the elements of G to V_i. Then G is isomorphic to $G_0 \times G_1 \times ... \times G_k$.*
2) *G_0 consists only of the identity transformation on V_0.*

3) *Each* G_i, $1 \leqslant i \leqslant k$, *is one of the groups described in Theorem 3.5.* G *is a Coxeter group iff* $V_0 = 0$.

The proof of Theorem 3.6 is identical with that of Theorem 2.7. We simply observe that we may now choose the V_i's to be mutually orthogonal.

2. The Computation of the Degrees for Real Finite Reflection Groups

Let G be a finite irreducible orthogonal reflection group acting on the n-dimensional Euclidean space R^n. Let F be a fundamental region as described in Theorem 3.3 and $R_1, \ldots, R_n$ the n reflections in the walls of F. We shall relate the degrees $d_1, \ldots, d_n$ of the basic homogeneous invariants to the eigenvalues of $R_1 \ldots R_n$. We first prove

Theorem 3.7. *Let* $\sigma(i)$ *be any permutation of* $1, \ldots, n$. *Then* $R_1 \ldots R_n$ *is conjugate to* $R_{\sigma(1)} \ldots R_{\sigma(n)}$

Proof. Observe that $R_1 (R_1 \ldots R_n) R_1 = R_2 \ldots R_n R_1$ so that all cyclic permutations yield conjugate transformations. We may also permute any two adjacent R_i's for which the corresponding walls are orthogonal, as the R_i's then commute. Theorem 3.7 will then follow from the following

Lemma 3.1. Let $p_1, \ldots, p_n$ be nodes of a tree T. Any circular arrangement of $1, \ldots, n$ can be obtained from a sequence of interchanges of pairs i, j which are adjacent on the circle and for which p_i, p_j are not linked in T.

Proof of Lemma 3.1. We proceed by induction, the result being obvious for $n = 1$ or 2. We may assume that p_n is an end node of the tree, i.e. it links to precisely one other node. We first rearrange $1, \ldots, n-1$ as we wish. To show that this can be done, we just consider the possibility - - - *inj* - - - where p_i, p_j are not linked. If p_i, p_n are not linked, then we interchange first i, n and then i, j, obtaining - - - *nji* - - -. If p_j, p_n are not linked, then we first interchange j, n and then j, i, obtaining - - - *jin* - - -. We may therefore arrange $1, \ldots, n-1$ in the desired order. Shifting n in one direction, which is permissible as n just fails to commute with one element, we obtain the desired arrangement of $1, \ldots, n$.

In view of Theorem 3.7, the eigenvalues of $R_1 \ldots R_n$ are independent of the order in which the R_i's appear. They are also independent of the particularly chosen F. For let F' be another fundamental region as described in Theorem 3.3. Then $F' = \sigma F$, $\sigma \in G$. The reflections in the walls of F'

are given by $R_i' = \sigma R_i \sigma^{-1}$, $1 \leqslant i \leqslant n$, so that $R_1' \ldots R_n' = \sigma R_1 \ldots R_n \sigma^{-1}$. The main result of the present section is the following

THEOREM 3.8 (Coleman [8]). *Let $R_1 \ldots R_n$ have order h. Let $\zeta = e^{2\pi i/h}$. The eigenvalues of $R_1 \ldots R_n$ are given by $\zeta^{(d_j - 1)}$, $1 \leqslant j \leqslant n$, the d_j's being the degrees of the basic homogeneous invariants of G.*

Theorem 3.8. was first obtained by Coxeter [7], who verified this fact for each group listed in Theorem 3.5. Coleman [8] supplied a general proof, using the fact that the number of reflections $= \frac{1}{2} nh$. This fact, which was at first known only by individual verification [7], was proven by Steinberg [20]. In view of Theorem 3.8, the numbers $m_j = d_j - 1$ are usually referred to as the exponents of the group G.

We begin by proving Steinberg's result, needed for the proof of Coleman's theorem. We require a preliminary lemma and employ the following terminology. Let $A = (a_{ij})$ be an $n \times n$ matrix with non-negative entries. We associate with A a graph $\mathscr{G}$ consisting of n nodes, connecting the nodes i, j iff $a_{ij} > 0$. A is said to be connected iff $\mathscr{G}$ is connected.

LEMMA 3.2. Let $A = (a_{ij})$ be a symmetric connected matrix. The largest eigenvalue λ of A is positive and a corresponding eigenvector e can be chosen all of whose entries are positive.

REMARK. The above is a special case of a theorem of Frobenius concerning the eigenvalues of matrices with non-negative entries [13]. Indeed the symmetry of A is not required. This extraneous assumption permits for a somewhat simpler proof and suffices for our purposes.

Proof. Let $Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ be the quadratic form associated with (a_{ij}). Then $\lambda = \operatorname{Max}_{\|x\|=1} Q(x) > 0$, where $\|x\|^2 = \sum_{i=1}^{n} x_i^2$. Choose $v = (v_1, \ldots, v_n)$, $\|v\| = 1$, so that $Q(v) = \lambda$ and let $e = (e_1, \ldots, e_n)$, where $e_i = |v_i|$, $1 \leqslant i \leqslant n$. Then $e_i \geqslant 0$, $1 \leqslant i \leqslant n$, and $\|e\| = 1$. As all $a_{ij} \geqslant 0$ and $\|e\| = 1$, we have $\lambda = Q(v) \leqslant Q(e) \leqslant \lambda$, so that $Q(e) = \lambda$. The latter implies $Ae = \lambda e$. It remains to show that each $e_i > 0$. Choose $e_j > 0$. Because of the connectivity assumption, we may choose $i_1, \ldots, j_r = j$ so that $a_{ij_1}, a_{j_1 j_2}, \ldots, a_{j_{r-1}, j}$ are all > 0. The relation $\lambda e_{j_{r-1}} = \sum_{k=1}^{n} a_{j_{r-1}, k} e_k$ shows that $e_{j_{r-1}} > 0$. Repeating this reasoning r times, we conclude that each $e_i > 0$.

THEOREM 3.9 (Steinberg [20]). *Let* $h =$ *order of* $R_1 \dots R_n$, $r =$ *number of reflections in* G. *Then* $r = \dfrac{nh}{2}$.

Proof. We may label the walls of the fundamental region F so that $W_1 \dots W_s$ are mutually perpendicular, and $W_{s+1}, \dots, W_n$ are mutually perpendicular (I.e. if the nodes corresponding to $W_1, \dots, W_s$ are black and those corresponding to $W_{s+1}, \dots, W_n$ are white, then each black node is linked only to white nodes and conversely). Let $E_1 = W_{s+1} \cap \dots \cap W_n$, $E_2 = W_1 \cap \dots \cap W_s$. Thus in terms of the dual basis $\{r_i'\}$, E_1 is the linear span of $r_1', \dots, r_s'$ and E_2 the linear span of $r_{s+1}', \dots, r_n'$. Let S $= R_{s+1} \dots R_n$, $T = R_1, \dots, R_s$ and denote the orthogonal complement of E_i, $i = 1, 2$, by $E_i^{\perp}$. The restriction of S to E_1, denoted by S_{E_1}, is the identity $r_{s+1}, \dots, r_n$ form a basis for $E_1^{\perp}$. Since they are orthogonal to each other, $R_i r_j = 0$ for $i \neq j$, $s + 1 \leqslant i, j \leqslant n$, so that $S_{E_1^{\perp}} = -$ identity. Similarly T_{E_2} $=$ identity, $T_{E_2^{\perp}} = -$ identity. We require the following

LEMMA 3.3. Let G_0 be the $n \times n$ matrix $((r_i, r_j))$ and I the $n \times n$ identity matrix. $I - G_0$ is connected. Thus, by Lemma 3.2, $I - G_0$ has a biggest positive eigenvalue λ and a corresponding eigenvector e with positive entries. Let $\sigma = \sum_{i=1}^{s} e_i r_i'$, $\tau = \sum_{i=s+1}^{n} e_i r_i'$ [1]). The plane π, determined by σ and τ, has non-trivial intersection with $E_1^{\perp}$ and $E_2^{\perp}$. It follows that S_π (T_π) is a reflection of π in the line through σ (τ).

Proof. The entries of $I - G_0$ are $\geqslant 0$, as $(r_i, r_j) \leqslant 0$ whenever $i \neq j$. The irreducibility of G is equivalent to saying that $I - G_0$ is connected. Let

$$G_0 = \begin{pmatrix} I & A \\ A' & I \end{pmatrix}, \quad G_0^{-1} = \begin{pmatrix} B & C \\ C' & D \end{pmatrix},$$

where A, C are $s \times n - s$ matrices (we use I to denote the identity matrix for various degrees; here degree $I = s$). The relations $r_i = \sum_{j=1}^{n} (r_i, r_j)\, r_j'$, $r_i' = \sum_{i=1}^{n} (r_i', r_j')\, r_j$, $1 \leqslant i \leqslant n$, show that $G_0{}^{-1} = ((r_i', r_j'))$. Since $G_0^{-1}\, G_0$ $= I$, we have

$$(3.1) \qquad BA + C = C' + DA' = 0$$

Let e^1 be the vector consisting of the first s components of e, e^2 the vector

[1]) Geometrically, the directions of σ, τ are those in E_1, E_2 which produce the smallest angle. To prove this, one solves this minimum problem by the method of multipliers. Lagrange's equations lead to (3.2.).

consisting of the last $n - s$ components of e. The equation $(I - G_0)\, e = \lambda e$ becomes

$$A\, e^2 + \lambda\, e^1 = A'\, e^1 + \lambda\, e^2 = 0\,. \tag{3.2}$$

(3.1), (3.2) imply

$$\lambda B\, e^1 - C\, e^2 = \lambda D\, e^2 - C'\, e^1 = 0\,. \tag{3.3}$$

Let $\sigma = \sum_{1=1}^{s} e_i\, r_i'$, $\tau = \sum_{i=s+1}^{n} e_i\, r_i'$. (3.3) may be rewritten as

$$\begin{aligned} r_i' \cdot (\lambda\sigma - \tau) &= 0, \quad 1 \leqslant i \leqslant s\,, \\ r_i' \cdot (\lambda\tau - \sigma) &= 0, \quad s + 1 \leqslant i \leqslant n\,. \end{aligned} \tag{3.4}$$

The vectors $\lambda\,\sigma - \tau$, $\lambda\,\tau - \sigma$ are $\neq 0$ and in π. (3.4) states that $\lambda\,\sigma - \tau \in E_1^{\perp}$, $\lambda\,\tau - \sigma \in E_2^{\perp}$. Since $\sigma \in E_1$, $\sigma' = \lambda\,\sigma - \tau \in E_1^{\perp}$, we have $S(\sigma) = \sigma$, $S(\sigma') = -\sigma'$. I.e. S_π is a reflection in the line through σ. Similarly, T_π is a reflection in the line through τ.

We now return to the proof of Theorem 3.9. Let H be the subgroup generated by S, T. H_π is the group generated by S_π, T_π. Let

$$F_0 = \{v \mid v = x\,\sigma + y\,\tau, x, y > 0\} = F \cap \pi\,.$$

F_0 is a fundamental region for H_π. For let $\gamma \in H$, $\gamma_\pi \neq I$. Then $\gamma \neq I$ and we have $\gamma_\pi\, F \cap F = \gamma\, F \cap F \cap \pi = \Phi$. R_π is a rotation of π through twice the angle between σ and τ. We show that ord $R_\pi = h$. For let ord $R_\pi = k$. Since $R^h = I$, $R_\pi^h = I$, we have $k \leqslant h$. Choose $p \in F_0$. $R^k(p) = R_\pi^k(p) = p$ so that $R^k\, F \cap F \neq \Phi \Rightarrow R^k = I \Rightarrow h \leqslant k$. Thus $h = k$. It follows that F_0 is an angular wedge of angular width $\dfrac{2\pi}{h}$ and H_π is a dihedral group of order $2h$. The h transforms of σ are contained in precisely $(n-s)$ r.h.'s. The h transforms of τ are contained in precisely s r.h.'s. Every r.h. of G has a non-trivial intersection with π. Since each of the transforms of F_0 is contained in a chamber of G and each chamber is free of r.h.'s, these r.h.'s meet π only at the transforms of σ and τ. Counting the r.h.'s at the transforms of σ and τ, we obtain the count $h\,s + h\,(n-s) = h\,n$. Each r.h. is however counted twice, as it intersects π in a line and thus meets two of the σ and τ transforms. Hence $r = \dfrac{h\,n}{2}$.

As a by product of the above proof, we obtain the following result required to establish Theorem 3.8.

THEOREM 3.10. *$\zeta = e^{2\pi i/h}$ is an eigenvalue of R. Corresponding to ζ, we may choose an eigenvector v not lying in any r.h.* (Note: if v is complex, then v is said to lie in the r.h. π iff $L(v) = 0$, $L(x) = 0$ being the equation of π).

Proof. Assume first that the R_i's are labeled as in the proof of Theorem 3.9; i.e. the walls $W_1, \ldots, W_s$ are mutually perpendicular as are also $W_{s+1}, \ldots, W_n$. Let π be the plane of Lemma 3.3. We choose two orthonormal vectors v_1, v_2 in π such that v_1 is not contained in any r.h. of G and

$$(3.5)\qquad \begin{aligned} R(v_1) &= \cos\frac{2\pi}{h}v_1 + \sin\frac{2\pi}{h}v_2 \\ R(v_2) &= -\sin\frac{2\pi}{h}v_1 + \cos\frac{2\pi}{h}v_2 \end{aligned}$$

Let $v = v_1 - iv_2$. We conclude from (3.5) that $R(v) = e^{2i\pi/h}v$. Thus v is an eigenvector corresponding to the eigenvalue $\zeta = e^{2i\pi/h}$. v is not in any r.h. of G as v_1 is not in any r.h. of G.

For an arbitrary labeling of indices, choose a permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ so that the above reasoning applies to $R' = R_{i_1} \ldots R_{i_n}$. By Theorem 3.7. $R = R_1 \ldots R_n = \sigma R' \sigma^{-1}$ for some $\sigma \in G$. Hence $R(\sigma v) = \zeta(\sigma v)$. Since the r.h.'s are permuted by σ, we conclude that σv is also not contained in any r.h. of G.

We also require

THEOREM 3.11. *1 is not an eigenvalue of R.*

REMARK. In Theorem 3.12 we obtain the characteristic equation of R, from which we may obtain Theorem 3.11. The following proof is shorter and avoids any explicit matrix representation for R.

Proof. Let π be the r.h. corresponding to the root r and σ the reflection in π. Then $v' = \sigma v$ becomes

$$(3.6)\qquad v' = v - 2(v,r)r$$

Suppose that $R_1 \ldots R_n v = v$, $\Leftrightarrow R_2 \ldots R_n v = R_1 v$. Repeated application of (3.6) shows that $R_2 \ldots R_n v = v + \lambda_2 r_2 + \ldots + \lambda_n r_n$, $\lambda_2, \ldots, \lambda_n$ being real numbers depending on v. Hence

$$(3.7)\qquad v + \lambda_2 r_2 + \ldots + \lambda_n r_n = v - 2(v, r_1) r_1$$

Since $r_1, \ldots, r_n$ are linearly independent we must have $(v, r_1) = 0$ $\Leftrightarrow R_1 v = v$, so that $R_2 \ldots R_n v = v$. Repeating the reasoning, we conclude $(v, r_i) = 0$, $1 \leqslant i \leqslant n$, $\Rightarrow v = 0$. Thus 1 is not an eigenvalue of $R_1 \dots R_n$.

We can now provide the

Proof of Theorem 3.8. Let $v_1, \dots, v_n$ be linearly independent eigenvectors of R with v_1 chosenas in Theorem 3.10; i.e. v_1 corresponds to the eigenvalue $\zeta = e^{2i\pi/h}$ and does not lie in any r.h. of G. Let $x_1, \dots, x_n$ be a coordinate system adapted to $v_1, \dots, v_n$. As $R^h = I$, all eigenvalues of R are h-th roots of I. By Theorem 3.11, 1 is not an eigenvalue of R. Hence the eigenvalues of R are $\zeta^{m_1}, \dots, \zeta^{m_n}$ where $m_1 = 1$ and $1 \leqslant m_1 \leqslant \dots \leqslant m_n = h - 1$, $1 \leqslant i \leqslant n$. R is given by $x_i' = \zeta^{m_i} x_i$, $1 \leqslant i \leqslant n$.

Let $I_1, \dots, I_n$ be a basic set of homogeneous invariants of G of respective degrees $d_1 \leqslant \dots \leqslant d_n$. By Theorem 2.5,

$$J = \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} \neq 0$$

off the r.h.'s of G. Hence $J \neq 0$ whenever $x = (x_1, 0, \dots, 0)$, $x_1 \neq 0$. It follows that there exists a permutation $j = j(i)$ of 1 to n such that

$$\frac{\partial I_i}{\partial x_j} (x_1, 0, \dots, 0) \neq 0$$

for $x_1 \neq 0$ and $1 \leqslant i \leqslant n$. This means that the $x_1^{d_i - 1}$ coefficient of

$$\frac{\partial I_i}{\partial x_j} \neq 0 \Rightarrow x_1^{d_i - 1} x_j$$

coefficient of $I_i \neq 0$, $1 \leqslant i \leqslant n$. Hence each $x_1^{d_i - 1} x_j$ is invariant under R. I.e.

$$(d_i - 1) + m_j \equiv 0 \pmod h, \quad 1 \leqslant i \leqslant n \tag{3.8}$$

Rewrite (3.8) as

$$d_i - 1 = (h - m_j) + \varepsilon_i h, \quad 1 \leqslant i \leqslant n \tag{3.9}$$

where each ε_i is an integer $\geqslant 0$. Let $m_j' = h - m_j$. The eigenvalues of R occur in pairs, so that the set of numbers $\{m_j'\}$ is identical with $\{m_j\}$. Summing both sides of (3.9) from $i = 1$ to $i = n$, we get

$$\sum_{i=1}^{n} (d_i - 1) = \sum_{j=1}^{n} m_j' + \Big(\sum_{i=1}^{n} \varepsilon_i\Big) h \tag{3.10}$$

By Theorem 2.2, $\sum_{i=1}^{n} (d_i - 1) = r$. Since

$$(3.11) \qquad \sum_{j=1}^{n} m_j' = \sum_{j=1}^{n} (h - m_j) = n\,h - \sum_{j=1}^{n} m_j',$$

we also have $\sum_{j=1}^{n} m_j' = \dfrac{n\,h}{2}$. We conclude from Theorem 3.9 that $\sum_{i=1}^{n} (d_i - 1) = \sum_{j=1}^{n} m_j'$. (3.10) shows that $\sum_{i=1}^{n} \varepsilon_i = 0 \Rightarrow \varepsilon_i = 0, 1 \leqslant i \leqslant n$. It follows from (3.9) that $d_i - 1 = m_i$, $1 \leqslant i \leqslant n$.

To make effective use of Coleman's Theorem, we need the explicit expression for the characteristic equation of R.

THEOREM 3.12 (Coxeter [5], p. 218). *The characteristic equation of* $R = R_1 \dots R_n$ *is given by*

$$(3.12) \qquad \begin{vmatrix} \dfrac{1+\lambda}{2} & \lambda a_{12} & \dots & \lambda a_{1n} \\ a_{21} & \dfrac{1+\lambda}{2} & \lambda a_{23} \dots & \lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n,n-1} & \dfrac{1+\lambda}{2} \end{vmatrix} = 0$$

where $a_{ij} = -\cos(\pi/p_{ij})$, $1 \leqslant i, j \leqslant n$.

Proof. Let $v = \sigma v'$ where σ is a reflection in the r.h. perpendicular to the root r.
Then

$$(3.13) \qquad v = v' - 2(v' \cdot r)\, r$$

We use (3.13) to obtain the matrix for R_j relative to the basis $r_1', \dots, r_n'$. Let $v = \sum_{i=1}^{n} x_i r_i'$, $v' = \sum_{i=1}^{r} x_i' r_i'$. Then $v' \cdot r_j = x_j'$, $r_j = \sum_{i=1}^{n} a_{ij} r_i'$. Substituting into (3.13), we get

$$(3.14) \qquad v = R_j v' \Leftrightarrow x_i = x_i' - 2a_{ij} x'_j,\ 1 \leqslant i \leqslant n$$

Let

$$v = R_1 v^{(1)},\ v^{(1)} = R_2 v^{(2)}, \dots, v^{(n-1)} = R_n v^{(n)}$$

so that $v = R_1 \dots R_n v^{(n)}$. Suppose that $v^{(j)} = \sum_{i=1}^{n} x_i^{(j)} r_i', 1 \leqslant j \leqslant n.$ We conclude from (3.14) that

$$\text{(3.15)} \qquad \begin{cases} x_i = x_i' - 2a_{i1}\, x_1' \\ x_i' = x_i'' - 2a_{i2}\, x_2'' \\ \dots\dots\dots\dots\dots, \; 1 \leqslant i \leqslant n \\ x_i^{(n-1)} = x_i^{(n)} - 2a_{in}\, x_n^{(n)} \end{cases}$$

Let $y_i = x^{(k)}, 1 \leqslant i \leqslant n$. For each i we rewrite (3.15) as

$$\text{(3.16)} \begin{cases} x_i' - x_i = 2a_{i1}\, y_1 \\ x_i'' - x_i' = 2a_{i2}\, y_2 \\ \dots\dots\dots\dots \\ y_i - x_i^{(i-1)} = 2a_{ii}\, y_i \end{cases} \qquad \text{(3.17)} \begin{cases} x_i^{(i+1)} - y_i = 2a_{i,i+1}\, y_{i+1} \\ x_i^{(i+2)} - x_i^{(i+1)} = 2a_{i,i+2}\, y_{i+2} \\ \dots\dots\dots\dots\dots\dots \\ x_i^{(n)} - x_i^{(n-1)} = 2a_{in}\, y_n \end{cases}$$

Adding up respectively the equations in (3.16), and (3.17), we obtain

$$\text{(3.18)} \qquad -x_i = \sum_{j=1}^{i-1} 2a_{ij}\, y_j + y_i, \; 1 \leqslant i \leqslant n$$

$$\text{(3.19)} \qquad x_i^{(n)} = \sum_{j=i+1}^{n} 2a_{ij}\, y_j + y_i, \; 1 \leqslant i \leqslant n$$

(3.18), (3.19) may be abbreviated as

$$\text{(3.20)} \qquad -x = Ay, x^{(n)} = A'\, y$$

where

$$\text{(3.21)} \quad A = \begin{bmatrix} 1 & & & & \\ 2a_{21} & & & & \\ . & 1 & & & \\ . & . & . & & \\ . & . & . & . & \\ 2a_{n1} & . & . & 2a_{n,\, n-1} & 1 \end{bmatrix}$$

the entries above the diagonal being zero.

Hence $x = -A\,(A')^{-1}\, x^{(n)}$, so that $-A\,(A')^{-1}$ is the matrix for $R = R_1 \dots R_n$ relative to the basis $r_1', \dots, r_n'$. The characteristic equation for R is thus given by

$$(3.22) \qquad | -A(A')^{-1} - \lambda I | = 0 \Leftrightarrow \left| \frac{A + \lambda A'}{2} \right| = 0$$

which is the same as (3.12).

We rewrite the characteristic equation in a more symmetric form. Suppose first that G is of type I. We label nodes of the graphs in diagram 3.2 from left to right as $1, \ldots, n$. Thus $a_{ij} = 0$ whenever $|j - i| > 1$. Multiplying first the i-th row of the determinant in (3.12) by $\lambda^{(i-1)/2}$, $1 \leqslant i \leqslant n$, then the j-th column by $\lambda^{-j/2}$, $1 \leqslant j \leqslant n$, we get

$$(3.23) \qquad \begin{vmatrix} \Lambda & & & & \\ & \cdot & & a_{ij} & \\ & & \cdot & & \\ & & & \cdot & \\ a_{ij} & & & \cdot & \\ & & & & \Lambda \end{vmatrix} = 0$$

where $\Lambda = \dfrac{\lambda^{1/2} + \lambda^{-1/2}}{2}$

If G is of type II, then the nodes on the principal chain are labeled from left to right as 1 to $n - 1$, the remaining node being labeled n. The n^{th} node is linked to the q^{th} node. Let $i' = i, j' = j$, $1 \leqslant i, j \leqslant n - 1$, and $i' = j' = q + 1$ whenever i or $j = n$. Multiply first the i-th row of the determinant in (3.12) by $\lambda^{\frac{i'-1}{2}}$, $1 \leqslant i \leqslant n$, then the j-th column by $\lambda^{-j'/2}$. We obtain again (3.23). We have proven

COROLLARY. *The characteristic equation of R is given by* (3.23).

We illustrate the use of Coleman's Theorem by computing the d_i's for the icosahedral group I_3. In this case the characteristic equation (3.23) becomes

$$(3.24) \qquad \begin{vmatrix} \Lambda & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \Lambda & -\cos\frac{\pi}{5} \\ 0 & -\cos\frac{\pi}{5} & \Lambda \end{vmatrix} = 0$$

The roots of (3.24) are readily computed to be $\zeta = e^{\frac{2\pi i}{10}}, \zeta^5, \zeta^9$. It follows from Coleman's Theorem that $d_1 = 2$, $d_2 = 6$, $d_3 = 10$.

3. Tabulation of the Degrees

Theorem 3.8 can be used to compute the degrees of the basic homogeneous invariants of G, in case G is an irreducible reflection group acting on R^n. This has been done in [7], and we tabulate these degrees below

Group	$d_1, \ldots, d_n$
A_n $(n \geqslant 1)$	$2, \ldots, n+1$
B_n $(n \geqslant 2)$	$2, 4, \ldots, 2n$
D_n $(n \geqslant 4)$	$2, 4, \ldots, n, \ldots, 2n-4, 2n-2$
H_2^n $(n \geqslant 5)$	$2, n$
E_6	2, 5, 6, 8, 9, 12
E_7	2, 6, 8, 10, 12, 14, 18
E_8	2, 8, 12, 14, 18, 20, 24, 30
F_4	2, 6, 8, 12
I_3	2, 6, 10
I_4	2, 12, 20, 30

We observe that in each case, $d_1 = 2$. This can be seen as follows. Suppose that there existed a homogeneous invariant $I(x)$ of degree 1. Since $I(\sigma x) = I(x)$ whenever $\sigma \in G$, the hyperplane $\{x \mid I(x) = 0\}$ would be a proper invariant subspace of G, contradicting that the latter is irreducible. Hence there are no homogeneous invariants of degree 1 and $d_1 \geqslant 2$. On the other hand, $\sum_{i=1}^{n} x_i^2$ is invariant under G as G is orthogonal. It follows that $d_1 = 2$, with corresponding invariant $I_1 = \sum_{i=1}^{n} x_i^2$.

In applying Theorem 3.8, we must find the roots of the characteristic equation (3.23). In some cases, this is a rather tedious computation. For the groups A_n, B_n, D_n H_2^n we can exhibit a basis of homogeneous invariants without the use of Theorem 3.8. We require

Theorem 3.13. *Let G be a finite reflection group acting on the n-dimensional vector space V over a given field k. Let $P_1, \ldots, P_n$ be homogeneous* *invariants of* G *of respective degrees* $k_1, \ldots, k_n$. $P_1, \ldots, P_n$ *form a basis for the invariants of* $G \Leftrightarrow k_1 \ldots k_n = |G|$ *and*

$$\Delta = \frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

Proof. By relabeling indices, we may assume $k_1 \leqslant \ldots \leqslant k_n$. The $\Rightarrow$ part of the theorem is contained in Theorems 1.2, 2.2, 2.3. Conversely, let $k_1 \ldots k_n = |G|$ and $\Delta \neq 0$. Thus $P_1, \ldots, P_n$ are algebraically independent. Let $I_1, \ldots, I_n$ be basic homogeneous invariants of respective degrees $d_1, \ldots, d_n$. Suppose $k_i = d_i$, $1 \leqslant i \leqslant i_0$, but $k_{i_0+1} < d_{i_0+1}$. Then $P_1, \ldots, P_{i_0+1}$ are polynomials in $I_1, \ldots, I_{i_0}$, implying that $P_1, \ldots, P_n$ are algebraically dependent, a contradiction. Hence $k_i \geqslant d_i$, $1 \leqslant i \leqslant n$. Since $\prod_{i=1}^{n} d_i = \prod_{i=1}^{n} k_i = |G|$, we must have $k_i = d_i$, $1 \leqslant i \leqslant n$.

Let $\delta_m = \dim \mathcal{J}_m$, $0 \leqslant m < \infty$, $\mathcal{J}_m$ being the space of homogeneous invariants of degree m. Then δ_m = number of non-negative integral solutions to $j_1 d_1 + \ldots + j_n d_n = m$. This number also equals the number of monomials $P_1^{j1} \ldots P_1^{jn}$ which are of degree m. The algebraic independence of $P_1, \ldots, P_n$ implies that these δ_m monomials are linearly independent over k. Thus $\mathcal{J}_m$ is spanned by these monomials for $0 \leqslant m < \infty$. We have shown that every homogeneous invariant is a polynomial in $P_1, \ldots, P_n$, so that the P_i's form a basis for the invariants of G.

We now obtain an explicit basis for the invariants of A_n, B_n, D_n, H_2^n. A_n: This group consists of the $(n+1)!$ permutations $x_i' = x_{\sigma(i)}$, $1 \leqslant i \leqslant n+1$, restricted to the subspace $V = \{x \mid x_1 + \ldots + x_{n+1} = 0\}$. We choose $x_1, \ldots, x_n$ as coordinates on V. Let $P_i = \sum_{j=1}^{n+1} x_j^{i+1}$, $1 \leqslant i \leqslant n$, where $x_{n+1} = -(x_1 + \ldots + x_n)$. P_i is a homogeneous invariant of degree $i+1$. We have $2 \cdot \ldots \cdot (n+1) = (n+1)! = |A_n|$.

We show that $\Delta \neq 0$. Now

$$\frac{\partial P_i}{\partial x_j} = (i+1)\, x_j^i - (i+1)\, x_{n+1}^i, \quad 1 \leqslant i,j \leqslant n .$$

Hence $\Delta = (n+1)!\, D$ where D is the $n \times n$ determinant whose (ij)-th entry $= x_j^i - x_{n+1}^i$. To evaluate D, we introduce the Vandermonde determinant

$$\begin{vmatrix} 1 & \ldots\ldots & 1 \\ x_1 & \ldots\ldots & x_{n+1} \\ x_1^n & \ldots\ldots & x_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j \leq n+1} (x_j - x_i)$$

Subtracting the $(n+1)$-th column from the first n columns, the above determinant is readily seen to equal $(-1)^n D$. Thus

$$(3.25) \qquad \Delta = (-1)^{n+2}(n+1)! \prod_{1 \leq i < j \leq n+1} (x_j - x_i) =$$

$$(n+1)! \prod_{1 \leq j \leq n} (x_j - x_i) \cdot \prod_{i=1}^{n} (x_i + s)$$

where $s = x_1 + \ldots + x_n$. (3.25) shows that $\Delta \neq 0$. We conclude that $d_1 = 2, \ldots, d_n = n + 1$.

B_n: Let $P_i = \sum_{j=1}^{n} x_j^{2i}$, $1 \leqslant i \leqslant n$. P_i is a homogeneous invariant of degree $2i$. We have $2 \cdot \ldots \cdot 2n = 2^n n! = | B_n |$. A computation shows that $\Delta = 2^n n! \prod_{i=1}^{n} x_i \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$. It follows that $d_1 = 2, \ldots, d_n = 2n$.

D_n: Let $P_1 = x_1 \ldots x_n$, $P_i = \sum_{j=1}^{n} x_j^{2(i-1)}$, $2 \leqslant i \leqslant n$. P_1 is a homogeneous invariant of degree n; P_i, $2 \leqslant i \leqslant n$, is a homogeneous invariant of degree $2(i-1)$. The product of the degrees $= n \cdot 2 \cdot 4 \cdot \ldots \cdot (2n-2) = 2^{n-1} n! = | D_n |$.

$$(3.26) \qquad \Delta = \begin{vmatrix} \dfrac{P_1}{x_1} & \ldots & \dfrac{P_1}{x_n} \\ 2x_1 & \ldots & 2x_n \\ \cdot & \ldots & \cdot \\ 2(n-1)x_1^{2n-3} & \ldots & 2(n-1)x_n^{2n-3} \end{vmatrix}$$

$$= 2^{n-1}(n-1)! \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \neq 0$$

It follows that $d_i, \ldots, d_n$ are identical with the numbers $2, 4, \ldots, n, \ldots, 2n-4, 2n-2$.

H_2^n: Let z be the complex coordinate $x_1 + i x_2$. H_2^n may be described as the group generated by the transformation $z \to \bar{z}$, $z \to \zeta z$, where $\zeta = e^{\frac{2\pi i}{n}}$. Let $P_1 = x_1^2 + x_2^2$, $P_2 = Re\, z^n$. P_1, P_2 are homogeneous invariants of respective degrees $2, n$. The product of these degrees $= 2n = | H_2^n |$. A computation yields

$$\frac{\partial (P_1, P_2)}{\partial (x_1, x_2)} = -2n\, Im\, z^n \neq 0 .$$

It follows that $d_1 = 2$, $d_2 = n$.

4. Solomon's Theorem

We present in this section another method for determining the degrees of the basic invariants, valid whenever the underlying field k has characteristic 0.

Theorem 3.14 (Solomon [18]). *Let G be a finite reflection group acting on the n-dimensional space V. Let g_r = number of elements of G which fix some r-dimensional subspace of V but do not fix a subspace of higher dimension. Let $d_1, \ldots, d_n$ be the degrees of the basic homogeneous invariants of G and set $m_j = d_j - 1$. Then*

$$(3.27) \qquad (t+m_1)\ldots(t+m_n) = g_0 + g_1 t + \ldots + g_n t^n$$

Equating the t^{n-1}-coefficients of both sides of (3.27), we obtain $g_1 = r = \sum_{i=1}^{n} m_i$. Setting $t = 1$ in (3.27), we obtain $\prod_{i=1}^{n} (m_i + 1) = \sum_{i=0}^{n} g_i = |G|$. Thus Theorem 3.14 generalizes Theorem 2.2.

To prove Theorem 3.14, we obtain an analog of Molien's formula for the invariant differential forms of G. We digress to a brief discussion of differential forms.

For $p > 0$, let $\omega = \sum_{i_1 < \ldots < i_p} r_{i_1 \ldots i_p}(x)\, dx_{i_1} \ldots dx_{i_p}$, where $r_{i_1 \ldots i_p}(x) \in k(x)$, the summation extending over all integer p-tuples satisfying $1 \leqslant i_1 < \ldots < i_p \leqslant n$. ω is called a differential p-form (or simply p-form). The elements of $k(x)$ are called the 0-forms. If $\eta = \sum_{i_1 < \ldots i_p} s_{i_1 \ldots i_p}(x)\, dx_{i_1} \ldots dx_{i_p}$ is another p-form, then we define

$$\omega + \eta = \sum_{i_1 < \ldots < i_p} (r_{i_1 \ldots i_p} + s_{i_1 \ldots i_p})\, dx_{i_1} \ldots dx_{i_p}.$$

Thus the p-forms constitute a vector space over $k(x)$ which we denote by $\mathscr{D}_p$. The elements $dx_{i_1} \ldots dx_{i_p}$ form a basis for $\mathscr{D}_p$, so that $\dim \mathscr{D}_p = \binom{n}{p}$, $0 \leqslant p \leqslant n$. We also define a multiplication between two forms as follows. Let $dx_i\, dx_j = -dx_j\, dx_i$; in particular $dx_i\, dx_i = 0$. The product $\omega\eta$ of any two forms ω, η is then obtained by the distributive law. We observe that for 1-forms, $\omega\eta = -\eta\omega$, so that $\omega\omega = 0$. It follows that $\mathscr{D}_p = 0$ for $p > n$. Finally, for any rational function r, we define the 1-form dr to be

$$\sum_{i=1}^{n} \frac{\partial r}{\partial x_i} dx_i .$$

It is then readily checked that for n rational functions, $r_1, ..., r_n$, we have

$$dr_1 \ldots dr_n = \frac{\partial (r_1, \ldots, r_n)}{\partial (x_1, \ldots, x_n)} dx_1 \ldots dx_n .$$

Let σ be a non-singular matrix with entries in k. We define

$$\sigma \omega = \sum_{i_1 < \cdots < i_p} r_{i_1 \cdots i_p} (\sigma^{-1} x)\, dx_{i_1} (\sigma^{-1} x) \ldots dx_{i_p} (\sigma^{-1} x)$$

Thus σ becomes a linear transformation on each $\mathscr{D}_p$, interpreting the latter as a vector space over k. Let k^n be the space of n-tuples with entries in k. If G is a group of linear transformations acting on k^n, then ω is said to be invariant under G provided $\sigma\omega = \omega$, $\forall\, \sigma \in G$.

We shall prove Theorem 3.14 describing the invariant differential forms with polynomial coefficients. G is assumed throughout to be a finite reflection group acting on k^n.

LEMMA 3.4. Let $I_1, ..., I_n$ be basic homogeneous invariants for G. Let

$$\Pi (x) = \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)} .$$

The polynomial $p(x)$ satisfies $\sigma p = (\det \sigma) p$, for every $\sigma \in G$ (in which case, we say p is skew) iff $p = \Pi i$ where i is a polynomial invariant under G.

Proof. Let $y = \sigma x$. Then

$$\Pi (x) = \frac{\partial (I_1 (y), \ldots, I_n(y))}{\partial (x_1, \ldots, x_n)} \tag{3.28}$$

$$= \frac{\partial (I_1 (y), \ldots, I_n (y))}{\partial (y_1, \ldots, y_n)} \det \sigma = \Pi (\sigma x) \det \sigma$$

which shows that Π is skew. Hence Πi is skew for every invariant polynomial i.

Conversely, let $p(x)$ be skew. Let π be an r.h. of G with equation $L(x) = 0$. By Lemma 2.2, we may choose $v \notin \pi$, so that v is a common eigenvector to all reflections in G with r.h. π. Choose $x = Ty$, $\det T \neq 0$, so that in the y coordinates the equation of π becomes $y_n = 0$ and v becomes $(0, ..., 0, 1)$. Let $q(y) = p(Ty)$. Let H be the subgroup of G which fixes π. By Lemma 2.2, H is a cyclic group. Let σ generate H and $h = \text{ord } H$. If ζ is the eigenvalue of σ which is a primitive h-th root of 1, then

$q(y_1, \ldots, y_{n-1}, \zeta y_n) = \zeta^{-1} q(y_1, \ldots, y_n)$. Writing $q = \sum q_i y_n^i$, the q_i's being polynomials in $y_1, \ldots, y_{n-1}$, we obtain

$$\Sigma q_i \zeta^{i+1} y_n^i = \Sigma q_i y_n^i \tag{3.29}$$

Equating coefficients in (3.29), we conclude $q_i = 0$ whenever $h \nmid i+1$. Thus $q_i = 0$ for $i < h-1 \Rightarrow y_n^{h-1} \,|\, q \Rightarrow L^{h-1} \,|\, p$. Repeating this argument for all r.h.'s of G and using Theorem 2.5, we conclude that $P = \Pi i$, where i is a polynomial. $\sigma i = \sigma P / \sigma \Pi = \dfrac{P}{\Pi} = i$ shows that i is invariant under G.

LEMMA 3.5. Let σ be a non-singular matrix with entries in k. Let $r \in k(x)$. Then $\sigma(dr) = d(\sigma r)$.

Proof. By definition

$$\sigma(dr) = \sum_{i=1}^{n} \frac{\partial r}{\partial x_i}(\sigma^{-1}x)\, dx_i(\sigma^{-1}x), \quad d(\sigma r) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(r(\sigma^{-1}x)\right) dx_i \tag{3.30}$$

Let $\sigma^{-1} = (a_{ij})$. Then $x_i(\sigma^{-1}x) = \sum_{j=1}^{n} a_{ij} x_j$ and $\dfrac{\partial x_i}{\partial x_j}(\sigma^{-1}x) = a_{ij}$.

Hence

$$dx_i(\sigma^{-1}x) = \sum_{j=1}^{n} a_{ij}\, dx_j \tag{3.31}$$

Applying the chain rule,

$$\frac{\partial}{\partial x_i}\left(r(\sigma^{-1}x)\right) = \sum_{j=1}^{n} \frac{\partial r}{\partial x_j}(\sigma^{-1}x)\, a_{ji} \tag{3.32}$$

Inserting (3.31), (3.32) into (3.30), we get $\sigma(dr) = d(\sigma r)$.

THEOREM 3.15. *Every invariant p-form with polynomial coefficients may be expressed uniquely as*

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p},\; a_{i_1 \ldots i_p} \in k[I_1, \ldots, I_n].$$

Proof. By Lemma 3.5, $\sigma(dI_k) = dI_k$, so that $dI_1, \ldots, dI_n$ are invariant forms. Since $\sigma(\omega\eta) = \sigma(\omega)\,\sigma(\eta)$ for any two forms ω, η, we conclude that $\sum_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p}$ is invariant whenever $a_{i_1 \ldots 1_p} \in k(I_1, \ldots, I_n)$.

We show that the $\binom{n}{p}$ forms $dI_{i_1} \ldots dI_{i_p}$ are linearly independent over $k(x)$, so that they form a basis for $\mathscr{D}_p$ over $k(x)$. Suppose that

$$\sum_{i_1 < \cdots < i_p} k_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p} = 0,\; k_{i_1 \ldots i_p} \in k(x).$$

Multiply this relation by $dI_{i_{p+1}} \dots dI_{i_n}$, where $i_{p+1}, \dots, i_n$ are the indices complementary to $i_1, \dots, i_p$. We obtain

$$k_{i_1 \dots i_p} dI_1 \dots dI_n = k_{i_1 \dots i_p} \Pi(x)\, dx_1 \dots dx_n = 0 \Rightarrow k_{i_1 \dots i_p} = 0$$

for all $i_1, \dots, i_p$. Hence the $\binom{n}{p}$ forms $dI_{i_1} \dots dI_{i_p}$ are linearly independent over $k(x)$. It follows that every p-form ω may be expressed uniquely as

$$\omega = \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p}, \; a_{i_1 \dots i_p} \in k(x).$$

If ω is invariant, then the group averaging argument shows that $a_{i_1 \dots i_p} \in k(I_1, \dots, I_n)$. Multiply both sides of the above relation by $dI_{i_{p+1}} \dots dI_{i_n}$. We get

$$\omega\, dI_{i_{p+1}} \dots dI_{i_n} = \pm \Pi\, a_{i_1 \dots i_p} dx_1 \dots dx_n \,. \tag{3.33}$$

Let ω be a p-form with polynomial coefficients. We conclude from (3.33) that $\Pi\, a_{i_1 \dots i_p}$ is a polynomial. Since $\Pi\, a_{i_1 \dots i_p}$ is skew, Lemma 3.4 implies that $\Pi\, a_{i_1 \dots i_p} = \Pi\, i$, i being an invariant polynomial. Hence $a_{i_1 \dots i_p} \in k[I_1, \dots, I_n]$ for all $i_1, \dots, i_p$, thus proving Theorem 3.11.

THEOREM 3.16. *Let $\sigma_p(x_1, \dots, x_n)$ be the p-th elementary symmetric function in $x_1, \dots, x_n$ (σ_0 is interpreted to be 1). Let $\omega_1(\gamma), \dots, \omega_n(\gamma)$ be the eigenvalues of γ, $\gamma \in G$. Then*

$$\frac{\sigma_p(t^{m_1}, \dots, t^{m_n})}{(1-t^{m_1+1}) \dots (1-t^{m_n+1})} \tag{3.34}$$

$$= \frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega_1(\gamma), \dots, \omega_n(\gamma))}{(1-\omega_1(\gamma)t) \dots (1-\omega_n(\gamma)t)}, \; 0 \leqslant p \leqslant n$$

REMARK. For $p = 0$, the above becomes formula (2.5) of Chapter II.

Proof. Let $\mathcal{D}_{pm}$ = space of p-forms whose coefficients are homogeneous polynomials of degree m. $\mathcal{D}_{pm}$ is a finite dimensional vector space over k. Let $\mathcal{I}_{pm}$ = space of invariant forms in $\mathcal{D}_{pm}$ and $d_{pm} = \dim \mathcal{I}_{pm}$. For $0 \leqslant p \leqslant n$, let $\mathfrak{p}_p(t) = \sum_{m=0}^{\infty} d_{pm} t^m$. We obtain two formulas for $\mathfrak{p}_p(t)$ by computing d_{pm} in two different ways. By Theorem 3.15, the differentials

$$I_1^{k_1} \dots I_n^{k_n} dI_{i_1} \dots dI_{i_p}, \; m = k_1(m_1+1) \dots + k_n(m_n+1)$$
$$+ m_{i_1} + \dots + m_{i_p},$$

form a basis for $\mathscr{J}_{pm}$, so that

$$\mathfrak{p}_p(t) = \frac{\sigma_p(t^{m_1}, \ldots, t^{m_n})}{(1-t^{m_1+1})\ldots(1-t^{m_n+1})} \tag{3.35}$$

Let $\tilde{k}$ = algebraic closure of k. Define $\tilde{\mathscr{D}}_{pm}$, $\tilde{\mathscr{J}}_{pm}$, analogously to $\mathscr{D}_{pm}$, $\mathscr{J}_{pm}$, replacing k by $\tilde{k}$. For $\gamma \in G$, γ acts both on $\mathscr{D}_{pm}$ and $\tilde{\mathscr{D}}_{pm}$. Let $(\mathrm{Tr}\ \gamma)_{pm}$ = trace of γ as a transformation on $\mathscr{D}_{pm}$ = trace of γ as a transformation on $\tilde{\mathscr{D}}_{pm}$. By Lemma 1.2

$$d_{pm} = \frac{1}{|G|} \sum_{\gamma \varepsilon G} (Tr\,\gamma)_{pm} \tag{3.36}$$

Choose T so that $T\sigma T^{-1} = D$, D being diagonal with diagonal entries $\omega_1(\gamma), \ldots, \omega_n(\gamma)$. The elements $x^a\, dx_{i_1} \ldots dx_{i_p}$, $|a| = m$ and $1 \leqslant i_1 < \ldots < i_p \leqslant n$, form a basis for $\tilde{\mathscr{D}}_{pm}$. Since

$$D(x^a dx_i \ldots dx_{i_p}) = [\omega(\gamma^{-1})]^a\, \omega_{i_1}(\gamma^{-1}) \ldots \omega_{i_p}(\gamma^{-1}), \tag{3.37}$$

we have

$$(TrD)_{pm} = \sum_{|a|=m} [\omega(\gamma^{-1})]^m\, \sigma_p(\omega(\gamma^{-1})) \tag{3.38}$$

(3.36), (3.38) yield

$$d_{pm} = \frac{1}{|G|} \sum_{\gamma \varepsilon G} \sum_{|a|=m} [\omega(\gamma)]^a\, \sigma_p[\omega(\gamma)] \tag{3.39}$$

so that

$$\begin{aligned} \mathfrak{p}_p(t) &= \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{r \varepsilon G} \sum_{|a|=m} [\omega(\gamma)]^a\, \sigma_p(\omega(\gamma))\, t^m \\ &= \frac{1}{|G|} \sum_{\gamma \varepsilon G} \frac{\sigma_p(\omega(\gamma))}{(1-\omega_1(\gamma)t)\ldots(1-\omega_n(\gamma)t)} \end{aligned} \tag{3.40}$$

(3.34) follows from (3.35) and (3.40).

We derive from (3.34) the following identity.

THEOREM 3.17. *For* $1 \leqslant p \leqslant n$,

$$\begin{aligned} &\sum_{i_1<\ldots<i_p} \frac{t^{m_{i_1}+\ldots+m_{i_p}}}{(1-t^{m_{i_1}+1})\ldots(1-t^{m_{i_p}+1})} \\ &= \frac{1}{|G|} \sum_{\gamma \varepsilon G} \sum_{i_1<\ldots<i_p} \frac{\omega_{i_1}(\gamma)\ldots\omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t)\ldots(1-\omega_{i_p}(\gamma)t)} \end{aligned} \tag{3.41}$$

Proof. One verifies readily, for $1 \leqslant p \leqslant n$, the identity

$$\sum_{i_1<\ldots<i_p} \frac{u_{i_1} \ldots u_{i_p}}{(1-u_{i_1}t) \ldots (1-u_{i_p}t)} \tag{3.42}$$

$$= \frac{h_{p1}(t)\,\sigma_1(u_1, \ldots, u_n) + \ldots + h_{pn}(t)\,\sigma_n(u_1, \ldots, u_n)}{(1-u_it) \ldots (1-u_nt)}$$

the u_i's being indeterminates and the h_{pi}'s being polynomials in t. Substitute for u_i, $\omega_i(\gamma)$ and average over the group. By Theorem 3.16, the group average becomes expression (3.42), u_i being replaced by t^{m_i}, thus proving (3.41).

We can now provide the

Proof of Theorem 3.14. Expand both sides of (3.41) in powers of $1 - t$ and equate the coefficients of $(1 - t)^{-p}$. For the left side this coefficient is

$$\sum_{i_1<\ldots<i_p} \frac{1}{(m_{i_1}+1) \ldots (m_{i_p}+1)}$$

Let γ be an element which fixes an r dimensional subspace, but does not fix a higher dimensional subspace. This means that precisely r of the eigenvalues of γ equal 1. γ contributes to the coefficient of $(1-t)^{-p}$ on the right side of (3.41) iff $r \geqslant p$, the contribution being $\binom{r}{p}$. It follows that for the right side, the $(1-t)^{-p}$ coefficient is $\frac{1}{|G|} \sum_{r=0}^{n} \binom{r}{p} g_r$. Since $\prod_{i=1}^{n} (m_i+1) = |G|$, we conclude that

$$\sum_{r=0}^{n} \binom{r}{p} g_r = \sum_{i_1<\ldots<i_{n-p}} (m_{i_1}+1) \ldots (m_{i_{n-p}}+1), \quad 1 \leqslant p \leqslant n \tag{3.43}$$

Note that for $p = 0$, (3.43) becomes $|G| = (m_1+1) \ldots (m_n+1)$. Hence (3.43) also holds for $p = 0$.

The left and right side of (3.43) equal respectively $\frac{1}{p\,!}$ (p-th derivative at $t = 1$) of $g_0 + \ldots + g_n t^n$, $(t+m_1) \ldots (t+m_n)$. Thus $(t+m_1) \ldots (t+m_n) = g_0 + \ldots + g_n t^n$.