

4. Decomposition of Finite Reflection Groups

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Let $J_i = \sum A_m y_n^m$, the A_m 's being polynomials in $y_1, \dots, y_{n-1}$. (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0$, $0 \leq m \leq h-1$. Since

$$\frac{\partial J_i}{\partial y_m} = \sum_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leq i \leq n. \right.$$

Hence

$$(2.21) \quad y_n^{h-1} \left| \frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)}, \right.$$

Since

$$\frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to $L^{h-1}(x) \mid J(x)$. It follows that if $L_i(x) = 0$, $1 \leq i \leq r$, are the r.h.'s for G , then $\prod_{i=1}^r L_i \mid J$. But $J, \prod_{i=1}^r L_i$ have the same degree r , so that $J = c \prod_{i=1}^r L_i$ $c \neq 0$.

4. DECOMPOSITION OF FINITE REFLECTION GROUPS

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

DEFINITION 2.3. Let the group G act on V . G is said to be reducible iff there exists a proper subspace W invariant under G ; i.e. $\sigma w \in W$ for $\sigma \in G, w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2$, V_1 and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

THEOREM 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V . If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V . Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition

$v = v_1 + v_2, v_i \in V_i (i=1, 2)$. Let $\eta v = v_2$ and set $\tau = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \eta \sigma^{-1}$.

τ satisfies the following:

- i) $\tau \sigma = \sigma \tau, \sigma \in G$. For $\sigma \tau = \frac{1}{|G|} \sum_{\sigma_1 \in G} \sigma \sigma_1 \eta (\sigma \sigma_1)^{-1} \sigma = \tau \sigma$
- ii) $\tau v_1 = 0, v_1 \in V_1$. For $\sigma^{-1} v_1 \in V_1, \sigma \in G$, so that $\eta \sigma^{-1} v_1 = 0 \Rightarrow \tau v_1 = 0$
- iii) $(1-\tau) v \in V_1, v \in V, 1$ denoting the identity of G . For $(1-\eta) v \in V_1$, so that $(1-\eta) \sigma^{-1} v \in V_1 \Rightarrow \sigma (1-\eta) \sigma^{-1} v \in V_1, \sigma \in G$. It follows that $(1-\tau) v = \frac{1}{|G|} \sum_{\sigma \in G} \sigma (1-\eta) \sigma^{-1} v \in V_1$.

Let $V'_2 = \tau V$. V'_2 is invariant under G as $\sigma(\tau v) = \tau(\sigma v)$. For any $v, v = \tau v + (1-\tau) v$. It follows from iii) that $V = V_1 + V'_2$. ii), iii) imply $\tau(1-\tau) = 0 \Leftrightarrow \tau = \tau^2$. Hence $\tau v'_2 = v'_2$ for $v'_2 \in V'_2$. Let $v_1 + v'_2 = 0$, where $v_1 \in V_1, v'_2 \in V'_2$. Applying τ to both sides, we get $v'_2 = 0$ and so $v_1 = 0$. Hence $V = V_1 \oplus V'_2$.

Repeated application of Maschke's Theorem yields the

COROLLARY. *Let G be a finite group acting on the finite-dimensional vector space V . Then $V = V_1 \oplus \dots \oplus V_s$, the V_i 's being invariant subspaces of V and G acting irreducibly on each V_i .*

For finite reflection groups, we have

THEOREM 2.7. *Let G be a finite reflection group acting on V . There exists a decomposition $V = V_1 \oplus \dots \oplus V_s$ into invariant subspaces such that :*

- 1) *Let $G_i = G|_{V_i}$ = group of restrictions of elements of G to V_i . Then G is isomorphic to $G_1 \times \dots \times G_s$*
- 2) *Each $G_i, 1 \leq i \leq s$, is a reflection group acting irreducibly on V_i .*

Proof. By the corollary to Theorem 2.6, there exists a decomposition $V = V_1 \oplus \dots \oplus V_s$, the V_i 's being invariant subspaces and G_i irreducible for $1 \leq i \leq s$. We label the V_i 's so that $V_1, \dots, V_r$ are 1-dimensional and $G|_{V_i} = \text{identity}$.

By the remark following Definition 2.1, for each reflection σ there exists an eigenvector $v \in V - \pi, \pi$ being the r.h. for σ . Call v a root of G . We have

$$(2.22) \quad \dim(V_i + \pi) + \dim(V_i \cap \pi) = \dim V_i + \dim \pi.$$

If $V_i \not\subset \pi$, then $V_i + \pi = V$ and we conclude from (2.22) that $\dim V_i = \dim (V_i \cap \pi) + 1$. I.e. $V_i \cap \pi$ is a hyperplane in V_i and $\sigma|_{V_i}$ a reflection on V_i . Choose $u \in V_i - \pi$ so that u is an eigenvector of σ . u is a multiple of the root v , so that $v \in V_i$. Thus $\sigma|_{V_i}$ is a reflection of V_i if $v \in V_i$, and the identity if $v \notin V_i$. Furthermore, each root v is in some V_i , $r + 1 \leq i \leq s$, otherwise the corresponding reflection σ would have been the identity.

Let $\tilde{G}_i =$ subgroup generated by those reflections whose roots are in V_i , $1 \leq i \leq s$. It is readily checked that $G = \tilde{G}_1 \times \dots \times \tilde{G}_s$, $G_i = \tilde{G}_i|_{V_i}$. If $\sigma \in \tilde{G}_i$ and $\sigma|_{V_i} =$ identity then $\sigma =$ identity. The mapping $\sigma \rightarrow \sigma|_{V_i}$ is thus an isomorphism from $\tilde{G}_i$ onto G_i .

THEOREM 2.8. *Let G be a finite reflection group acting on V and decompose V as in Theorem 2.7. Every polynomial invariant under G is a polynomial in the invariant polynomials of $G_1, \dots, G_s$.*

Proof. For each $v \in V$, write $v = v_1 + \dots + v_s$, $v_i \in V_i$. By Theorem 2.7, for each $\sigma \in G$, we may write $\sigma v = \sigma_1 v_1 + \dots + \sigma_s v_s$, $\sigma_i \in G_i$. For any polynomial function $p(v)$ on V , we have $p(v) = \sum_{i=1}^N p_{i1}(v_1) \dots p_{is}(v_s)$ where $p_{ij}(v_j)$ is a polynomial function on V_j . If $p(v)$ is invariant under G , then

$$(2.23) \quad p(v) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma v) = \sum_{i=1}^N I_{i1}(v_1) \dots I_{is}(v_s)$$

where

$$(2.24) \quad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \in G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of G_j .

CHAPTER III

THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case G is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where k is the real field and has the advantage of providing an effective method for computing the