3. Homology and Cohomology of Groups

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.04.2024**

mination of the products for dimensions not greater than 2 (omitting considerations of torsion). In each case, the formulas are in terms of the fundamental group G, and pure group-theoretic constructions, but with geometric meanings."

3. Homology and Cohomology of Groups

As Whitney's review does suggest, Hopf's paper had immediate influence. His description of the second integral homology group of a group G was followed by four independent studies, two of which described the higher homology groups $H_n(G, \mathbf{Z})$ and two the higher cohomology group $H^n(G, A)$ for an abelian group A or, more generally, for a G-module A. Each of these papers explicitly recognizes the starting point provided by the paper of Hopf. In chronological order, these four studies are as follows:

Eilenberg and Mac Lane [1942] had been applying methods of group extensions to the universal coefficient theorem in cohomology, so they knew the group $\mathrm{Ext}(G, A)$ of all abelian extensions of the abelian group A by the abelian group G. They knew that a representation of G as F/R, with F and R free abelian, would give an exact sequence

$$0 \to \hom(G, A) \to \hom(F, A) \to \hom(R, A) \to \mathrm{Ext}(G, A) \to 0$$

(though they expressed this fact differently, writing $\mathrm{Ext}(G, A)$ as a suitable quotient of $\hom(R, A)$). Moreover, they had heard of the Schur multiplicator through Mac Lane's work on class field theory. Furthermore, Eilenberg was very familiar with homotopy groups. Hence, as soon as they saw the Hopf 1942 paper, they decided that more group extensions must be hidden in Hopf's G_1^*, and they set about to find out how.

On April 7, 1943 Eilenberg and Mac Lane submitted to the *Proceedings* of the National Academy of Sciences an announcement "Relations between homology and homotopy groups". Given a group G, they constructed a chain complex $K(G)$, whose second homology group is exactly Hopf's group G_1^*. Their complex $K(G)$—now called the Eilenberg-Mac Lane complex $K(G, 1)$—had as generators in dimension n the cells $[x_1, \ldots, x_n]$ for entries $x_i \in G$, with boundary

$$\partial[x_1, \ldots, x_n] = [x_2, \ldots, x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1, \ldots, x_i x_{i+1}, \ldots, x_n] + (-1)^n [x_1, \ldots, x_{n-1}] .$$

(If we use these cells to generate a free G-module and add the operator x_1 to the first boundary terms, this is just the bar resolution.) For any dimension n, the cohomology groups $H(K(G), A)$ with coefficients in an abelian group A were called the cohomology group of G with coefficients A. The essential topological result reads

THEOREM. If a space X is arcwise connected and has vanishing homotopy groups

$$\Pi_n(X) = 0 \qquad 1 < n < r$$

then the n-dimensional cohomology of X is given by

$$H^n(X, A) \cong H^n(\Pi_1(X), A), \qquad n < r,$$
$$\Lambda^r(X, A) \cong H^r(\Pi_1(X), A), \qquad n = r.$$

Here $\Lambda^r(X, A)$ is the subgroup of the r dimensional cohomology group H^r consisting of those cohomology classes which annihilate the spherical subgroup $S^r(X)$—consisting of those integral homology classes which can be represented by continuous images of spheres (as in the case of S^2 in Hopf's theorem for a polyhedron K).

In this paper there was also a corresponding theorem for the homology of X. It was formulated for the singular homology of an arbitrary space X, rather than for a complex, as in the work of Hopf. This is essentially a technical change, made possible by the fact that Eilenberg [1944] in the meantime had carried out the definitive formulation of singular homology. The essential fact was the same: The algebraic formulation of the influence of Π_1. In the simplest case: For an arcwise connected space X which is aspherical ($\Pi_n(X) = 0$ for all $n > 1$), the homology and cohomology of X depend only on the fundamental group $\Pi_1(X)$ and can be expressed algebraically as the homology and cohomology of the group $\Pi_1(X)$.

This paper of Eilenberg-Mac Lane also establishes briefly the corresponding result for an arcwise connected space X with exactly one non-vanishing homotopy group $\Pi_n(X)$. (An "Eilenberg-Mac Lane space") again by way of a suitable chain complex $K(\Pi_n(X), n)$ which represented, in algebraic form, a "minimal" singular complex of such a space X.

The next paper chronologically was Hopf's paper (communicated April 1, 1944 to *Commentarii*) "Uber die Bettische Gruppen, die zu einer beliebige Gruppe gehören". This paper describes the homology groups of a group G with coefficients in a G-module J. First form an exact sequence (Hopf didn't call it that or write it so!)

$$0 \leftarrow J \leftarrow X^0 \leftarrow X^1 \leftarrow \ldots \leftarrow X^n \leftarrow \tag{3}$$

of free G-modules, that is, of free modules over the integral group ring $P = Z[G]$. Regard the group $\mathbf{Z}$ of integers as a G-module with trivial action. Then the homology of the complex

$$\mathbf{Z} \otimes_G X^0 \leftarrow \mathbf{Z} \otimes_G X^1 \leftarrow \ldots \leftarrow \mathbf{Z} \otimes_G X^n \leftarrow$$

is independent of the choice of the exact sequence (3). Its n^{th} homology group is called the n^{th} Betti group $H_n(G, J)$. Moreover, Hopf proves that these algebraically defined groups are the homology groups of an arcwise connected aspherical space with fundamental group G, exactly as in the case above.

In formulating these facts we have changed Hopf's technique slightly. He didn't speak of exact sequences, because he hadn't yet had "the word" (which was invented about this time by Eilenberg and Steenrod for their axiomatic treatment of homology). He didn't explicitly use the tensor product $\mathbf{Z} \otimes_G X$ but instead used a quotient of X, which amounted to taking the augmentation map $\mathbf{Z}(G) \to \mathbf{Z}$, the corresponding short exact sequence $I(G) \rightarrowtail \mathbf{Z}(G) \to \mathbf{Z}$ and tensoring it with X. These are wholly minor differences. The essential fact is that Hopf had a clear formulation of the use of a *free* resolution and of the comparison theorem for two such resolutions (ideas not present in the Eilenberg-Mac Lane theorem cited earlier). Moreover his argument for his result replaced the aspherical space X by its universal covering space. Hence his use of different resolutions is clearly derived from the topological fact that *different* subdivisions of the same acyclic space (the universal covering space) will yield the same equivariant cohomology.

The third paper is by H. Freudenthal "Der Einfluss der Fundamental Gruppe auf die Bettischen Gruppen", published in the *Annals of Mathematics* in April 1946 and submitted there some time before July 29, 1945 (probably smuggled out of the Netherlands during the war). The paper was based on the *first* Hopf paper; because of the difficulty of communication during the war, its author did not know of the work of Eilenberg-Mac Lane, nor of the 1944 paper by Hopf discussed just above. Freudenthal's paper again uses free resolutions to define the homology and cohomology groups of G, and establishes essentially the same theorems relating these groups to the groups of an arcwise connected space aspherical in low dimensions. His use of free resolutions is again clearly a reflection of the properties of universal covering spaces.

The fourth paper, by Beno Eckmann "Der Cohomologie Ring einer beliebigen Gruppe", was communicated to the Commentarii on December 4, 1945. At that time Eckmann knew of both papers of Hopf, but did not know the papers of Eilenberg-Mac Lane or of Freudenthal. Given the group G and a ring J (with unit) his paper describes the cohomology ring $H^*(G, J)$ of G with coefficients in J. (In present terminology, this is a graded ring composed of various homology groups $H^n(G, J)$, each in its appropriate dimension n.) These cohomology groups are described by suitable cocycles and the ring structure is given by a suitable product, modeled after the Čech-Whitney cup product in topology. The main theorem again asserts that an arcwise connected space X with fundamental group G and aspherical in dimensions less the n has its cohomology ring in these dimension given by $H^*(G, J)$.

Eckmann also describes his cohomology group $H^n(G, J)$ as the cohomology of a chain complex K_G. This complex K_G is identical to the Eilenberg-Mac Lane complex $K(G, 1)$—but differently described. For Eilenberg-Mac Lane the n-cells of $K(G, 1)$ are the n-tuples $[x_1, \ldots, x_n]$ of elements $x_i \in G$. For Eckmann they are n-tuples $[y_1, \ldots, y_n]$; the translation is $y_i = x_1 \ldots x_i$ for $i = 1, \ldots, n$.

The ring structure, clearly formulated in Eckmann's paper, had been noted in the other three papers—as a cup product structure in Eilenberg-Mac Lane and as a (intersection) structure in Hopf and Freudenthal.

Thus we have four substantially independent discoveries of the same facts: The algebraic definition of the n-dimensional homology (or cohomology) of a group G and its identification with the homology (or cohomology) of a suitably aspherical space with fundamental group G. All four papers are based on (and inspired by) the original paper of Hopf for $n = 2$. The fact that there were as many as four substantially independent discoveries is undoubtedly due to the sharply limited international communication during wartime. This unintended experiment does go to show that the first Hopf paper was a breakthrough, recognized as such. Because of its structure, more development was possible—and was sure to be carried out.

Such a breakthrough itself must depend on previous ideas and developments. In this case the breakthrough involved a continuation of ideas both from algebra and from topology; we now turn to examine these.