7. Computation of SIv

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for all r. Therefore $ISf(0) = 0$ and hence $f(0) = 0$ by (8). If this result is applied to $(T_y^{-1})^* f$ it follows that $f(y) = 0$ for arbitrary y, so that f is indeed identically zero.

7. Computation of SIv

It is easy to show that $S_{ij,hk}(y) = [S\gamma_{ij,\cdot}(y)]_{hk}$ is a Calderon-Zygmund kernel for any choice of the indices; in other words, it is homogeneous of degree $-n$, and its mean-value over the unit sphere is 0. If $v \in L^p$, $1 < p < \infty$, it follows by the Calderon-Zygmund theory that the principal value

$$\text{pr. v.} \int_B v_{ij}(x)\, S_{ij,hk}(x-y)\, dx$$

exists almost everywhere, and that it is the limit in $L^p(B)$ of the corresponding truncated integrals. In view of (7) it follows that the integral

$$\Gamma v(y)_{hk} = \int_B v_{ij}(x)\, \Gamma_{ij,hk}(x,y)\, dx \tag{9}$$

will also exist as a principal value almost everywhere. One finds, however, that the remainder in (7) makes it possible to assert merely that the principal value is a limit in $L^{p'}$ for any $p' < p/n$. In these circumstances it is natural to assume that $v \in L^p(B)$ for all $p \geqq 1$.

Theorem 2. *If $v \in L^p(B)$ with $p > n$, then $SIv \in L^{p'}(B)$ for all $1 \leqslant p' < p/n$, and*

$$SI\,v = -\,b_n v + \Gamma v \tag{10}$$

where $b_n = 4\,\omega_n/(n+2)$ and Γv is defined by (9).

Proof. Let φ be an SM_n-valued test-function. The definition of SIv as a distribution leads to the following formal computation:

$$\int_B SI\,v(y)_{hk}\,\varphi(y)_{hk}\,dy = -\int_B Iv(y)_k S^*\varphi(y)_k\,dy$$
$$= -\int_B S^*\varphi(y)_k\,dy \int_B v_{ij}(x)\,\gamma_{ij,k}(x,y)\,dx$$
$$= -\int_B v_{ij}(x)\,dx \int_B S^*\varphi(y)_k\,\gamma_{ij,k}(x,y)\,dy$$
$$= -\int_B v_{ij}(x)\,dx\,\Big[b_n\,\varphi_{ij}(x) - \int_B \varphi(y)_{hk}\,\Gamma_{ij,hk}(x,y)\,dy\Big].$$

The justification, by means of the Zygmund-Calderon theory, is routine, and (10) follows.

Taken together, Theorems 1 and 2 lead to a very striking result:

THEOREM 3. *An SM_n-valued function $v \in L^p(B)$, $p > n$, is of the form $v = Sf$ with $f = 0$ on $S(1)$ if and only if it satisfies the homogeneous integral equation $\Gamma v = -a_n v$ with $a_n = c_n - b_n = 2(n-2)(n+1)\omega_n / n(n+2)$.*

Indeed, if v is of this form, Theorem 1 implies $c_n f = -Iv$, hence $c_n v = -SIv$, and consequently $\Gamma v = (b_n - c_n) v$ by Theorem 2. Conversely, if $\Gamma v = -a_n v$ then $SIv = -c_n v$ by (10), and $f = Iv$ vanishes on $S(1)$.

The point of Theorem 3 is that the solvability of $Sf = v$ (with an extra condition on f) has been reduced to an integral equation.

THEOREM 4. *For any $v \in L^p(B)$, $p > n$, $S^* \rho\, [\Gamma v + a_n v] = 0$.*

Proof. Let f be a vector-valued test-function. Theorem 3 applies to Sf, and we obtain by use of Lemma 2

$$\int_B S^* \rho\, \Gamma v \cdot f\, dx = - \int_B \rho(x)\, \Gamma v(x)_{ij}\, Sf(x)_{ij}\, dx$$

$$= - \int_B \rho(x)\, Sf(x)_{ij}\, dx \int_B v(y)_{hk}\, \Gamma_{hk,ij}(y, x)\, dy$$

$$= - \int_B \rho(y)\, v(y)_{hk} dy \int_B Sf(x)_{ij}\, \Gamma_{ij,hk}(x, y)\, dx$$

$$= - \int_B \rho(y)\, v(y)_{hk}\, \Gamma Sf(y)_{hk} dy = a_n \int_B \rho(y)\, v(y)_{hk}\, Sf(y)_{hk} dy$$

$$= - a_n \int_B S^* \rho v \cdot f\, dy$$

and hence $S^* \rho\, \Gamma v = -a_n S^* v$.

THEOREM 5. *Every v which is in all $L^p(B)$ has a unique representation in the form $v = v' + v''$ where v' and v'' are in all $L^p(B)$ while v' is in the image of SI and v'' is in the kernel of $S^* \rho$.*

As a consequence of Theorems 3 and 4 the representation is given by

$$c_n v = - SIv + (\Gamma v + a_n v)\,.$$

It is unique, for if $SI = \Gamma v + a_n v$, then $S^* \rho\, SIv = 0$ so that Iv is harmonic and 0 on $S(1)$, hence identically zero.