2. EQUIVARIANT HOMOTOPY THEORY

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

similar. We work with the category of pointed G-spaces, i.e. G-spaces with fixed point. The suspension functor Σ and the loop-space functor Ω are then defined, also the binary functors wedge $\vee$ and smash $\wedge$. This corresponds, of course, to the category of sectioned G-bundles, i.e. G-bundles with cross-section, where these functors are also defined and commute with the principal functor. We prefer, however, to enlarge this to the category of ex-spaces — see § 4.

2. Equivariant homotopy theory

Let G be a topological group and let A_i $(i=1, 2)$ be a pointed G-space. The space of pointed G-maps $f: A_1 \to A_2$ is denoted by $M_G(A_1, A_2)$, and the set of pointed G-homotopy classes of pointed G-maps by $\pi_G(A_1, A_2)$. The class of the constant map $e: A_1 \to A_2$ is denoted by 0. In this context we reserve the symbol $\simeq$ for the relation of pointed G-homotopy.

Let A be a pointed G-space with base-point a_0, and let $p, q: A \to A \times A$ be given by

$$p(x) = (x, a_0), \quad q(x) = (a_0, x) \qquad (x \in A).$$

By a *Hopf G-structure* on A we mean a pointed G-map $m: A \times A \to A$ such that

$$mp \simeq 1 \simeq mq: A \to A;$$

given such a structure we refer to A as a *Hopf G-space*. For example, the reduced product space A_∞ (see [5]) of any pointed G-space A is an associative Hopf G-space [1]). If A_2 is a Hopf G-space then $\pi_G(A_1, A_2)$, for any pointed G-space A_1, obtains a natural binary operation with 0 as neutral element [2]).

If $m: A \times A \to A$ satisfies the conditions for a topological group then, as before, we describe A as a group G-space. If m satisfies these conditions up to pointed G-homotopy then we describe A as a *group-like G-space*. Note that $\pi_G(A_1, A_2)$ is a group when A_2 is group-like. This is so, in particular, when $A_2 = \Omega A_2'$ with standard Hopf G-structure for any pointed G-space A_2'. If A_2' itself is a Hopf G-space then the group is abelian, by the usual argument.

---

[1]) Under suitable conditions it can be shown, using Segal's theorem, that A_∞ has the same pointed G-homotopy type as $\Omega\Sigma A$.

[2]) Another application of Segal's theorem is to show, following Sugawara [14], that this binary set forms a loop, under suitable conditions, and hence a group when the Hopf G-structure is pointed G-homotopy associative.

The notions of *coHopf G-space* and *cogroup-like G-space* are defined in the obvious way. If A_1 is a coHopf G-space then $\pi_G(A_1, A_2)$, for any pointed G-space A_2, obtains a binary operation with 0 as neutral element; moreover $\pi_G(A_1, A_2)$ is a group when A_1 is cogroup-like. In particular ΣA_1 is cogroup-like, for any pointed G-space A_1, and the adjoint functor determines an isomorphism

$$\xi: \pi_G(\Sigma A_1, A_2) \to \pi_G(A_1, \Omega A_2)\,.$$

Using this we can define Whitehead products in equivariant homotopy theory as follows.

We say that a pointed G-space X is *well-based* if there exists a neighbourhood U of the basepoint x_0 in X such that

(i) x_0 is an equivariant deformation retract, rel x_0, of U, and

(ii) there exists an invariant map $u: X \to I$ such that $ux_0 = 1$ and $ux = 0$ for $x \notin U$.

Let A, B be well-based G-spaces. Then (cf. [12]) for any pointed G-space Y the equivariant form of the Puppe sequence

$$0 \to \pi_G(A \wedge B, \Omega Y) \overset{p^*}{\to} \pi_G(A \times B, \Omega Y) \overset{q^*}{\to} \pi_G(A \vee B, \Omega Y) \to 0$$

is exact. Here q denotes the inclusion and p the collapsing map. From given elements $\alpha' \in \pi_G(A, \Omega Y)$, $\beta' \in \pi_G(B, \Omega Y)$ we can obtain $\alpha'', \beta'' \in \pi_G(A \times B, \Omega Y)$ by precomposition with the structural maps of the product. Since the commutator $\alpha''^{-1} \,.\, \beta''^{-1} \,.\, \alpha'' \,.\, \beta''$ lies in the kernel of q^* there exists, by exactness, a (unique) element

$$<\alpha', \beta'> \in \pi_G(A \wedge B, \Omega Y)$$

with image this commutator. The *Samelson pairing*

$$\pi_G(A, \Omega Y) \times \pi_G(B, \Omega Y) \to \pi_G(A \wedge B, \Omega Y)$$

thus defined is bilinear, just as in [1], and has the property that

$$<\alpha', \beta'> \;=\; - \, T^* <\beta', \alpha'>\,,$$

where T denotes the switching map. The *Whitehead product* $[\alpha, \beta]$ of elements $\alpha \in \pi_G(\Sigma A, Y)$, $\beta \in \pi_G(\Sigma B, Y)$ is defined by

$$\xi\,[\alpha, \beta] \;=\; <\xi\alpha, \xi\beta>\,,$$

where ξ denotes the adjoint isomorphism. Clearly the Whitehead pairing

$$\pi_G(\Sigma A, Y) \times \pi_G(\Sigma B, Y) \to \pi_G(\Sigma(A \wedge B), Y)$$

thus defined is bilinear and has the property that

(2.1) $$[\alpha, \beta] = -(\Sigma T)^* [\beta, \alpha] .$$

It is a straightforward exercise, as in the ordinary theory, to show that the Whitehead pairing vanishes if Y is a Hopf G-space, and hence vanishes under suspension. Moreover the suspension ΣZ of a compact well-based G-space Z is a Hopf G-space if and only if the Whitehead square

$$w(\Sigma Z) \in \pi_G(\Sigma(Z \wedge Z), \Sigma Z)$$

of the identity vanishes.

It should also be noted that the Jacobi identity holds for Samelson products and hence for Whitehead products, by an equivariant version of the argument given by G. W. Whitehead [16]. Specifically, consider the permutations

$$B \wedge C \wedge A \xrightarrow{\sigma} A \wedge B \wedge C \xrightarrow{\tau} C \wedge B \wedge A ,$$

where A, B, C are suspensions of pointed G-spaces. Let

$$\alpha \in \pi_G(\Sigma A, Y) , \quad \beta \in \pi_G(\Sigma B, Y) , \quad \gamma \in \pi_G(\Sigma C, Y) ,$$

where Y is a pointed G-space. Then the relation

(2.2) $$[\alpha, [\beta, \gamma]] + (\Sigma\sigma)^* [\beta, [\gamma, \alpha]] + (\Sigma\tau)^* [\gamma, [\alpha, \beta]] = 0$$

holds in the group $\pi_G(\Sigma(A \wedge B \wedge C), Y)$.

3. Some examples

We need to begin by discussing briefly some relations between the category of G-spaces and the category of pointed G-spaces, as follows. Given spaces A, B we denote points of the join $A*B$ by triples (a, b, t) where $a \in A$, $b \in B$, $t \in I$, so that (a, b, t) is independent of a when $t = 0$, of b when $t = 1$. A basepoint $b_0 \in B$ determines a basepoint $(a, b_0, 0)$ in $A*B$. If A, B are G-spaces we make $A*B$ a G-space with action

$$(a, b, t) g = (ag, bg, t) \qquad (g \in G) .$$

Note that $A*B$ is pointed if B is. When $B = S^0$, with trivial action, then $A*B = \tilde{\Sigma} A$, the unreduced suspension [2]).

[1]) This differs by an automorphism from the normal definition.
[2]) We regard this as an identification space of the cylinder, in the usual way.