2. A group theoretical proof of the Hurwitz-Radon Theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **23 (1977)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

One may show dy direct matrix arguments that, if $m \geqslant 2$, then $c = 0$ if n is odd (corresponding to the absence of a non-singular vector field on S^{n-1}) and that all even values of c occur if n is even. The question whether odd values of c occur reduces to the question whether $c = 1$ occurs and this in turn leads to the consideration of the Hurwitz-Radon Theorem (see Section 2). Eckmann uses the then existing, scanty knowledge of homotopy groups of Stiefel manifolds to obtain special results (when $q \neq n - 1$)—we would do the same today, but would benefit from our more extensive knowledge.

Indeed, Eckmann himself returned to the question 24 years later when he lectured at a Battelle Rencontre in Seattle [67; 1968]. By this time, of course, Adams had proved his celebrated *Hopf Invariant One Theorem* and the properties of topological K-theory had been substantially developed. Eckmann performed the significant feat of explaining the theory, and its applications—to systems of linear equations, to the existence of (generalized) vector products in $\mathbf{R}^n$, to the parallelizability of spheres, and to the existence of almost-complex structures on spheres—of explaining all this to an audience dominated by theoretical physicists! What testimony to his clarity—and courage!

2. A GROUP THEORETICAL PROOF OF THE HURWITZ-RADON THEOREM

Immediately following the work discussed above, Eckmann produced [9; 1943] a truly beautiful proof of the celebrated theorem on the composition of quadratic forms. The problem is to determine, given n, those values of p such that there exist n bilinear forms $z_1, \ldots, z_n$ of the variables $x_1, \ldots, x_p$; $y_1, \ldots, y_n$, with complex coefficients, such that the identity

$$(2.1) \qquad (x_1^2 + \ldots + x_p^2)(y_1^2 + \ldots + y_n^2) = z_1^2 + \ldots + z_n^2$$

holds. As formulated by Radon in 1923, the solution is the following. Let $n = u \cdot 2^{4\alpha+\beta}$ with u odd and $0 \leqslant \beta \leqslant 3$. Then we can find $z_1, \ldots, z_n$ to satisfy (2.1) if and only if $p \leqslant 8\alpha + 2^\beta$. Actually, Radon considered forms with real coefficients, but Eckmann showed explicitly in his proof that a solution of (2.1) for forms with complex coefficients implies a solution for forms with real coefficients. Eckmann's proof is based on the classical theory of (complex) representations of finite groups, together with certain particular results, due to Frobenius and Schur, relating complex to real representations. Before outlining Eckmann's proof, let me quote Eckmann's own remarks justifying his sortie into this well-established field, since it contains a key statement of his approach to what may be called the *systematics* of mathematical exposition. Eckmann wrote:

"Es erscheint aus folgenden Gründen nicht überflüssig, zu den Beweisen von Hurwitz und Radon noch einen dritten hinzuzufügen: einmal ist unser Beweis einfacher und kürzer—dafür operiert er aber mit weniger elementaren Begriffen und Sätzen; ferner sind die Methoden von Hurwitz wie auch von Radon ad hoc konstruiert und liegen außerhalb der sonst in der Algebra üblichen, während wir die Frage in die wohlbekannten Gedankengänge der Darstellungstheorie einordnen, wo sie als schönes Beispiel für die Anwendung allgemeiner Sätze erscheint." [1])

Once again the problem is first replaced by a matrix problem (this step is, of course, common to all three proofs). Thus we seek p complex orthogonal $n \times n$ matrices $A_1, \ldots, A_p$, such that $A_k A_l' + A_l A_k' = 0$, $l \neq k$. By normalizing we may instead seek $p-1$ complex orthogonal $n \times n$ matrices $A_1, \ldots, A_{p-1}$ such that

$$(2.2) \quad A_k^2 = -I, A_k A_l = -A_l A_k, k, l = 1, 2, \ldots, p-1, k \neq l.$$

Ignoring for the moment the orthogonality condition, Eckmann considers the abstract group G, generated by $(a_1, a_2, \ldots, a_{p-1}, \varepsilon)$, subject to the relations

$$(2.3) \quad \varepsilon^2 = 1, a_k^2 = \varepsilon, a_k a_l = \varepsilon a_l a_k, k, l = 1, 2, \ldots, p-1, k \neq l,$$

and investigates complex representations of G of degree n whereby ε is represented by $-I$. The order of G is 2^p; if $p = 2$, $G = \mathbf{Z}/4$ and the problem is trivial, so we assume $p \geqslant 3$. G has 2^{p-1} non-equivalent representations of degree 1; by counting conjugacy classes it follows that, if p is odd, G has an irreducible representation of degree $f > 1$ and, if p is even, then G has 2 irreducible representations of degrees $f, f' > 1$. It is easy to see (from standard theorems of representation theory) that $f = 2^{\frac{p-1}{2}}$ if p is odd, and that $f = f' = 2^{\frac{p-2}{2}}$ if p is even. Moreover ε will be represented by $-I$ in these irreducible representations, since it cannot be

[1]) "It seems, on the following grounds, not to be superfluous to add a third proof to those of Hurwitz and Radon: on the one hand, our proof is simpler and shorter, although it employs less elementary ideas and theorems; further, the methods of Hurwitz and Radon were constructed in ad hoc fashion and lie outside the domain of standard algebra, whereas we set the problem in the familiar framework of representation theory, *where it serves as a beautiful example for the application of general theorems*". (my italics).

represented by I. Thus the degree n of an arbitrary representation of G of the required kind is given by

(2.4) $$n = m \cdot 2^{\frac{p-1}{2}}, \ p \text{ odd}; \quad n = m \cdot 2^{\frac{p-2}{2}}, \ p \text{ even}.$$

It remains to determine which of those representations are equivalent to an orthogonal representation—these will also, according to Frobenius-Schur, be equivalent to orthogonal *real* representations. Corresponding to an irreducible representation D of G, one computes $S = \sum_{g \in G} \chi(g^2)$, where χ is the character function. Then D is real-equivalent if and only if $S > 0$; D is equivalent to its complex conjugate $\bar{D}$ if $S < 0$; and D is not equivalent to $\bar{D}$ if $S = 0$. By a very beautiful application of the elementary theory of complex numbers, Eckmann used this criterion to show that the given irreducible representations of G (whereby ε is represented by $-I$) are real-equivalent (that is, orthogonal-equivalent) if $p \equiv 7, 0, 1 \bmod 8$, and not otherwise. If $p \equiv 3, 4, 5 \bmod 8$ they are equivalent to their complex conjugates; if $p \equiv 2, 6 \bmod 8$ they are not. One may immediately deduce the degrees of real-irreducible real representations of G, and hence show that for a given $n = u \cdot 2^t$, with u odd, the maximum value of p such that there exists a real (orthogonal) representation of G of degree n, in which ε is represented by $-I$, is given by the rule:

$$\begin{aligned} t &= 4\alpha \quad\ \ : p = 8\alpha + 1 \\ t &= 4\alpha + 1: p = 8\alpha + 2 \\ t &= 4\alpha + 2: p = 8\alpha + 4 \\ t &= 4\alpha + 3: p = 8\alpha + 8 \,. \end{aligned}$$

This is the Hurwitz-Radon Theorem. Today we know that, when translated into the language of vector fields on spheres, the Hurwitz-Radon number $p - 1$ provides an upper bound on the number of vector fields on S^{n-1} even without the linearity condition; this was, of course, proved by Adams exploiting the techniques of topological K-theory.

3. Complexes with operators

Here perhaps I trespass somewhat on Saunders MacLane's territory. But I do want to exemplify a characteristic feature of Eckmann's thought, whereby he passes freely to and fro between topology and algebra, gener-